ALGEBRAIC GEOMETRY IN FIRST ORDER LOGIC

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Abstract:

In every variety of algebras Θ we can consider its logic and its algebraic geometry. In the previous papers geometry in equational logic, i.e., equational geometry has been studied. Here we describe an extension of this theory towards the First Order Logic (FOL). The algebraic sets in this geometry are determined by arbitrary sets of FOL formulas. The principal motivation of such generalization lies in the area of applications to knowledge science.

In this paper the FOL formulae are considered in the context of algebraic logic.

With this aim we define special Halmos categories. These categories in the algebraic geometry related to FOL play the same role as the category of free algebras Θ^0 play in the equational algebraic geometry.

The paper consists of three parts. Section 1 is of introductory character. The first part (sections 2–4) contains background on algebraic logic in the given variety of algebras Θ. The second part is devoted to algebraic geometry related to FOL (sections 5–7). In the last part (sections 8–9) we consider applications of the previous material to knowledge science.

Contents

1. Algebra and logic

1.1 Multi-sorted algebra. Keeping in mind applications, throughout the paper the term algebra means *multi-sorted*, i.e., not necessarily one-sorted, algebra. We fix a set of sorts Γ. In the considered varieties Θ this set is finite, but it need not to be finite in general. We meet infinite Γ in the next section.

For every algebra $G \in \Theta$ we write

$$G = (G_i, i \in \Gamma).$$

The set of operations Ω is called the *signature of algebras* in Θ. Every symbol $\omega \in \Omega$ has a type $\tau = \tau(\omega) = (i_1, \dots, i_n; j), i, j \in \Gamma$. An operation of type τ is a mapping

$$G_{i_1} \times \ldots \times G_{i_n} \to G_j.$$

All operations of the signature Ω satisfy some set of identities. These identities define the variety Θ of Γ-sorted Ω-algebras. Let us consider homomorphisms and free algebras in Θ . A homomorphism of algebras in Θ has the form

$$\mu = (\mu_i, i \in \Gamma): G = (G_i, i \in \Gamma) \to G' = (G'_i, i \in \Gamma).$$

Here $\mu_i: G_i \to G'_i$ are mappings of sets, coordinated with operations in Ω. A congruence $Ker\mu = (\mathrm{Ker}\mu_i, i \in \Gamma)$ is the kernel of a homomorphism μ.

We consider multi-sorted sets $X = (X_i, i \in \Gamma)$ and the corresponding free in Θ algebras

$$W = W(X) = (W_i, i \in \Gamma).$$

A set X and a free algebra W can be presented as the free union of all X_i and all W_i, respectively.

Every (multi-sorted) mapping $\mu : X \to G$ is extended up to a homomorphism $\mu : W \to G$. Denote the set of all such μ by $\mathrm{Hom}(W, G)$. If all X_i are finite, we treat this set as an affine space. Homomorphisms $\mu: W \to G$ are points of this space.

For the given $G = (G_i, i \in \Gamma)$ and $X = (X_i, i \in \Gamma)$ we can consider the set

$$G^X = (G_i^{X_i}, i \in \Gamma).$$

It is the set of mappings

$$\mu = (\mu_i, i \in \Gamma)\colon X \to G.$$

There is a natural bijection $\mathrm{Hom}(W, G) \to G^X$. More information about multi-sorted algebras can be found in [Pl1].

Now let us turn to the models. Fix some set of symbols of relations Φ. Every $\varphi \in \Phi$ has its type $\tau = \tau(\varphi) = (i_1, \ldots, i_n)$. A relation, corresponding to φ, is a subset in the Cartesian product $G_{i_1} \times \ldots \times G_{i_n}$. Denote by $\Phi\Theta$ the class of models (G, Φ, f), where $G \in \Theta$, and f is a interpretation of the set Φ in G. As for homomorphisms of models, they are homomorphisms of the corresponding algebras which are coordinated with relations.

1.2 Logic.

In the sequel for the sake of clearness of the exposition we sometimes "forget" that algebras are multi-sorted in general and use one-sorted language. We consider logic in the given variety Θ. For every finite X, there is a logical signature

$$L = L_X = \{\vee, \wedge, \neg, \exists x, \ \ x \in X\},$$

where X is $\bigcup\limits_{i \in \Gamma} X_i$ for a finite Γ. We consider the set (more precisely, the L-algebra) of formulas $L\Phi W$ over the free algebra $W = W(X)$. This algebra is an L-algebra of formulas of FOL over the given Θ, Φ, and X.

First we define the atomic formulas. They are equalities of the form $w \equiv w'$, with $w, w' \in W$ of the same sort and the formulas $\varphi(w_1, \ldots, w_n)$, where $w_i \in W$, and all w_i are positioned according to the type $\tau = \tau(\varphi)$ of the relations φ and to the sorts. The set of all atomic formulas we denote by $M = M_X$. Define $L\Phi W$ to be the absolutely free L_X-algebra over M_X.

Let us consider another example of an L_X-algebra.

Given $W = W(X)$ and $G \in \Theta$, denote by $\mathrm{Bool}(W, G)$ the Boolean algebra $\mathrm{Sub}(\mathrm{Hom}(W, G))$ of all subsets in $\mathrm{Hom}(W, G)$. Define the action of quantifiers in $\mathrm{Bool}(W, G)$. Let A be a subset in $\mathrm{Hom}(W, G)$ and $x \in X_i$ be a variable of the sort i. Then $\mu\colon W \to G$ belongs to the set $\exists x A$ if there exists $\nu\colon W \to G$ in A such that $\mu(y) = \nu(y)$ for every $y \in X$ of the sort $j, j \neq i$, and for every $y \in X_i$, $y \neq x$. Thus we get an L-algebra $\mathrm{Bool}(W, G)$.

Now let us define a mapping

$$\mathrm{Val}_f^X\colon M_X \to \mathrm{Bool}(W,G),$$

where f is a model (the subject of knowledge), which realizes the set Φ in the given G. If $w \equiv w'$ is an equality of the sort i, then we set:

$$\mu : W \to G \in \mathrm{Val}_f^X(w \equiv w') = \mathrm{Val}^X(w \equiv w')$$

if $\mu_i(w) = \mu_i(w')$ in G. Here the point μ is a solution of the equation $w \equiv w'$. If the formula is of the form $\varphi(w_1, \dots, w_n)$, then

$$\mu \in \mathrm{Val}_f^X(\varphi(w_1, \dots, w_n))$$

if $\varphi(\mu(w_1), \dots, \mu(w_n))$ is valid in the model (G, Φ, f). Here $\mu(w_j) = \mu_{i_j}(w_j)$, i_j is the sort of w_j. The mapping Val_f^X is uniquely extended up to the L-homomorphism

$$\mathrm{Val}_f^X\colon L\Phi W \to \mathrm{Bool}(W,G).$$

Thus, for every formula $u \in L\Phi W$ we defined its value $\mathrm{Val}_f(u)$ in the model (G, Φ, f), which is an element in $\mathrm{Bool}(W,G)$.

Every formula $u \in L\Phi W$ can be viewed as an equation in the given model. Then a point $\mu\colon W \to G$ is the solution of the "equation" u if $\mu \in \mathrm{Val}_f(u)$.

1.3 Geometrical Aspect.

In the L-algebra of formulas $L\Phi W$, $W = W(X)$, we consider its various subsets T. On the other hand, we consider subsets A in the affine space $\mathrm{Hom}(W,G)$, i.e., elements of the L-algebra $\mathrm{Bool}(W,G)$. For each model (G, Φ, f) and for these T and A we establish the following *Galois correspondence* between sets of formulas in L-algebra of formulas $L\Phi W$ and sets of points in the space $\mathrm{Hom}(W,G)$:

$$T^f = A = \bigcap_{u \in T} \mathrm{Val}_f(u),$$
$$A^f = T = \{u | A \subset \mathrm{Val}_f(u)\}.$$

Here $A = T^f$ is a locus of all points satisfying the formulas from T. We regard T also as a system of "equations", where each "equation" is represented by a formula u from T. Every set A of such kind is said to be an *algebraic set* (or closed set, or algebraic variety), determined for the given model. We define *knowledge* as

$$(X, T, A, (G, \Phi, f)).$$

Here T is a *description of knowledge* and (G, Φ, f) is a *subject of knowledge*. $A = T^f$ is a *content of knowledge*, represented as an algebraic variety, X is a *place of knowledge* (the place, where the knowledge is situated). A set A can be regarded also as a relation between elements of G derived from equalities and relations of the basic set Φ. The relation $A = T^f$ belongs to the multi-sorted set

$$G^X = \{G_i^{X_i}, \ \ i \in \Gamma\}.$$

A set T of the form $T = A^f$ for some A is called an *f-closed set*. For an arbitrary T we have its closure $T^{ff} = (T^f)^f$ and for every $A \subset \mathrm{Hom}(W, G)$ we have the closure $A^{ff} = (A^f)^f$.

It is easy to understand that the following rule takes place:

A formula v belongs to the set T^{ff} if and only if the formula

$$(\bigwedge_{u \in T} u) \to v$$

holds in the model (G, Φ, f).

If the set T is infinite then the corresponding formula is called *infinitary*.

We want to study knowledge with different, changing "places of knowledge" X. In this case one should consider different $W = W(X)$, different "spaces of knowledge" $\mathrm{Hom}(W(X), G)$, and different $L\Phi W(X)$.

Free in Θ algebras $W(X)$ with finite X are the objects of the category, denoted by Θ^0. Morphisms of this category $s\colon W(X) \to W(Y)$ are arbitrary homomorphisms of algebras. The category Θ^0 is a full subcategory in the category Θ.

We intend to build a new category related to the first order logic for the given Θ. This category will play for the FOL the role similar to that of the category of free algebras Θ^0 for the equational logic. With this end we turn from pure logic to algebraic logic. Such a transition will allow us to associate description of knowledge with its content in a more interesting way. The sets of the type $T = A^f$ also look more natural.

2. Algebraic logic

2.1 The main idea. Algebraic logic deals with algebraic structures, related to various logical structures which correspond to different logical calculi. For example,

Boolean algebras are associated with classical propositional logic, Heyting algebras are associated with non-classical propositional logic, Tarski cylindric algebras and Halmos polyadic algebras are associated with FOL.

Every logical calculus assumes that there are formulas of the calculus, axioms of logic and rules of inference. On this basis a syntactical equivalence of formulas compatible with their semantical equivalence is defined. The transition from pure logic to algebraic logic is grounded on treating logical formulas up to a certain equivalence. We call the corresponding classes the *compressed formulas*. This transition leads to various special algebraic structures, in particular to the structures mentioned above.

Logical calculi are usually associated with some infinite set of variables. Denote such a set by X^0. In our situation it is a multi-sorted set $X^0 = (X^0_i, i \in \Gamma)$. Keeping in mind theory of knowledge and its geometrical aspect we will use a system of all finite subsets $X = (X_i,\ i \in \Gamma)$ of X^0 instead of this infinite universum. This gives rise to multi-sorted logic and multi-sorted algebraic logic. Every formula has a definite type (sort) X. Denote the new set of sorts by Γ^0. It is a set of all finite subsets of the initial set X^0.

2.2 Halmos Categories and multi-sorted Halmos algebras.

Fix some variety of algebras Θ. This means that a finite set of sorts Γ, a signature $\Omega = \Omega(\Theta)$ related to Γ, and a system of identities $Id(\Theta)$ are given.

Define Halmos categories for the given Θ.

First, for the given Boolean algebra B we define its existential quantifiers [HMT]. Existential quantifiers are the mappings $\exists\colon B \to B$ with the conditions:

1) $\exists 0 = 0$,

2) $a < \exists a$,

3) $\exists(a \wedge \exists b) = \exists a \wedge \exists b$, $0, a, b \in B$.

The universal quantifier $\forall\colon B \to B$ is defined dually:

1) $\forall 1 = 1$,

2) $a > \forall a$,

3) $\forall(a \vee \forall b) = \forall a \vee \forall b$.

Let B be a Boolean algebra and X a set. We say that B is a *quantifier X-algebra* if a quantifier $\exists x\colon B \to B$ is defined for every $x \in X$ and for every two elements

$x, y \in X$ the equality $\exists x \exists y = \exists y \exists x$ holds.

One may consider also *quantifier X-algebras B with equalities* over $W(X)$. In such algebras, to each pair of elements $w, w' \in W(X)$ of the same sort it corresponds an element $w \equiv w' \in B$ satisfying the conditions

1) $w \equiv w$ is the unit in B,

2) $(w_1 \equiv w_1' \wedge \ldots \wedge w_n \equiv w_n') < (w_1 \ldots w_n \omega \equiv w_1' \ldots w_n' \omega)$ where ω is an operation in Ω and everything is compatible with the type of operation.

Now we will give the general definition of the Halmos category for the given Θ, which will be followed by examples.

Halmos category H for an arbitrary finite $X = (X_i, i \in \Gamma)$ fixes some quantifier X-algebra $H(X)$ with equalities over $W(X)$. $H(X)$ are the objects of H.

The morphisms in H correspond to morphisms in the category Θ^0. Every morphism s_* in H has the form

$$s_* \colon H(X) \to H(Y),$$

where $s \colon W(X) \to W(Y)$ is a morphism in Θ^0.

We assume that

1) The transitions $W(X) \to H(X)$ and $s \to s_*$ yield a (covariant) functor $\Theta^0 \to H$.

2) Every $s_* \colon H(X) \to H(Y)$ is a Boolean homomorphism.

3) The coordination with the quantifiers is as follows:

3.1) $s_1 \exists x a = s_2 \exists x a, \quad a \in H(X)$, if $s_1 y = s_2 y$ for every $y \in X,\ y \neq x$.

3.2) $s \exists x a = \exists (sx)(sa)$ if $sx = y \in Y$ and $y = sx$ not in the support (see 4.2) of sx', $x' \in X,\ x' \neq x$.

4) The following conditions describe coordination with equalities

4.1) $s_*(w \equiv w') = (sw \equiv sw')$ for $s \colon W(X) \to W(Y)$, $w, w' \in W(X)$ are of the same sort.

4.2) $s^x_w a \wedge (w \equiv w') < s^x_{w'} a$ for an arbitrary $a \in H(X), x \in X, w, w'$ of the same sort with x in $W(X)$, and $s^x_w \colon W(X) \to W(X)$ is defined by the rule: $s^x_w(x) = w, sy = y, y \in X,\ \ y \neq x$.

This completes the definition of the Halmos category for a given Θ.

Now we turn to the definition of multi-sorted Halmos algebras. Suppose that all finite X are subsets of some infinite universum X^0.

The set of sorts we denote by Γ^0. It is the set of all finite subsets in the fixed X^0.

For every $X \in \Gamma^0$ we consider the signature

$$L_X = \{\vee, \wedge, \rceil; \exists x, x \in X\}$$

Let L be the union of all $L_X, X \in \Gamma^0$, V be the set of all equalities $w \equiv w', w, w'$ of the same sort in $W(X)$. The equalities are considered as nullary operations. Let $S = S_\Theta$ be the set of all $s : W(X) \to W(Y)$, $X, Y \in \Gamma^0$. Denote $\tilde{L} = L \cup S_\Theta \cup V$, (cf. 1.2). The multi-sorted signature $\tilde{L}$ we consider as a new signature for the FOL and we deal with the algebras in this signature, which have the form $H = \{H(X), X \in \Gamma^0\}$. Here $H(X)$ is an algebra in the signature L_X, equipped by a unary operations $s \colon H(X) \to H(Y)$ for $s : W(X) \to W(Y)$ and nullary operations from V. Let us take the subvariety of all Halmos algebras in the variety of all H and denote it by Hal_Θ.

The identities of Hal_Θ are the following:

1) The Boolean identities for every $H(X)$,

2) The identities, making every $\exists x : H(X) \to H(X)$ an existential quantifier,

3) The identities, making every $H(X)$ a quantorian X-algebra with equalities,

4) The identities, making every $s : H(X) \to H(Y)$ a Boolean homomorphism,

5) The identities, correlating the operations of the type s with quantifiers and equalities,

6) The identities, making the system of all $H(X)$ and all s a category and $\Theta^0 \to H$ a functor.

Algebras $H = \{H(X), X \in \Gamma^0\}$ with the operations from the extended signature $\tilde{L}$ and subject to the identities above are called *multi-sorted Halmos algebras*. They constitute the variety Hal_Θ of all multi-sorted Halmos algebras in Θ. It is evident that if all X are subsets of some universum X^0 then every Halmos category is a multi-sorted Halmos algebra, and every multi-sorted Halmos algebra can be treated as a Halmos category. However, one should take into account that in this situation the notions of subalgebra and subcategory are different. The same situation holds for homomorphisms. Speaking about homomorphisms we mean the homomorphisms of Halmos algebras.

2.3 The category and algebra $\mathrm{Hal}_\Theta(G)$.

We fix an algebra G in the variety Θ. For an arbitrary finite set $X = \{x_1, \cdots, x_n\}$ and the free in Θ algebra $W = W(x_1, \cdots, x_n)$ we consider the set of homomorphisms $\mathrm{Hom}(W, G)$ as an affine space. The points in this space are homomorphisms $\mu : W \to G$. There is a bijection

$$\mathrm{Hom}(W, G) \to G^{(n)}$$

given by

$$\mu \to (\mu(x_1), \cdots, \mu(x_n)).$$

Let now Bool (W, G) be the Boolean algebra of all subsets in $\mathrm{Hom}(W, G)$

$$Bool(W, G) = Sub(\mathrm{Hom}(W, G)).$$

We define quantifiers $\exists x : Bool(W, G) \to Bool(W, G),\ x \in X$ in this algebra in the following way: $\mu \in \exists x A \Leftrightarrow \exists \nu \in A$ such that $\mu(y) = \nu(y)$ for $y \in X,\ y \neq x$. Here A is an arbitrary set in $\mathrm{Hom}(W, G)$. The axioms of the existential quantifiers and equalities are fulfilled (see, for example, [Pl 1]). We have here a quantorian X-algebra with equalities $Bool(W, G)$ and we set

$$\mathrm{Hal}_\Theta(G)(X) = Bool(W(X), G).$$

Consider now morphisms of the category $\mathrm{Hal}_\Theta(G)$. Take some $s : W(X) \to W(Y)$ in Θ^0. We have

$$\tilde{s} : \mathrm{Hom}(W(Y), G) \to \mathrm{Hom}(W(X), G),$$

$\tilde{s}(\nu) = \nu s$ for arbitrary $\nu : W(Y) \to G$. If A is a subset in $\mathrm{Hom}(W(X), G)$, then $\nu \in s_* A = sA$ iff $\tilde{s}(\nu) \in A$. Hence, we get the mapping

$$s_* : Bool(W(X), G) \to Bool(W(Y), G),$$

which is a Boolean homomorphism. It is easy to check that such s_* are correctly correlated with quantifiers and equalities ([Pl1]). Thus, the Halmos category $\mathrm{Hal}_\Theta(G)$ is defined. If, further, we confine ourselves by the sets X in the given X^0, then we come to the Halmos algebra $\mathrm{Hal}_\Theta(G)$.

Theorem 1. *The algebras* $\mathrm{Hal}_\Theta(G)$ *with different* $G \in \Theta$ *generate the variety of Halmos algebras* Hal_Θ*.*

The proof of this theorem will be given in 4.3.

Note that to every s_* there corresponds the conjugate mapping

$$s^* : Bool(W(Y).G) \to Bool(W(X), G),$$

where for every $A \subset \mathrm{Hom}(W(Y), G)$ the set s^*A is the $\tilde{s}$-image of the set A. From now on we assume that such a conjugate mapping $s^* : H(Y) \to H(X)$ always exists in Halmos algebras and categories.

3. The category and the algebra of formulas $\mathrm{Hal}_{\Phi\Theta}$

3.1 The definition of the algebra $\mathrm{Hal}_\Theta(\Phi)$. Compressed formulas..

For some reasons we will use also the notation $\mathrm{Hal}_\Theta(\Phi)$ for $\mathrm{Hal}_{\Phi\Theta}$. We start from the algebra of pure formulas and will construct $\mathrm{Hal}_{\Phi\Theta} = \mathrm{Hal}_\Theta(\Phi)$ as an algebra of the special compressed formulas.

Fix the set X^0 and a set Φ of symbols of relations. This Φ can be also an empty set. For the given X the atomic formulas of the type (sort) X are equalities $w \equiv w'$, $w, w' \in W(X)$ and formulas $\varphi(w_1, \cdots, w_n)$ with n-ary $\varphi \in \Phi$ and $w_1, \cdots, w_n \in W(X)$. We denote the set of all such formulas by M_X. We have also the multi-sorted set M with $M(X) = M_X, X \in \Gamma^0$.

Further we proceed from the signature $\tilde{L}$, defined earlier for FOL. Let H be the multi-sorted (Γ^0-sorted) absolutely free algebra in $\tilde{L}$ over the set of all atomic formulas M. It is an algebra of pure formulas of FOL in Θ. Every formula $u \in H$ has some special sort X.

Factorizing H by the identities of the variety Hal_Θ we get the free in Hal_Θ algebra $\tilde{H}$. Here $\tilde{H}$ is a free Halmos algebra over the same set of atomic formulas M.

Every formula in H has a finite record in the signature $\tilde{L}$ through atomic formulas. The same is valid for the formulas in $\tilde{H}$.

We consider relations for the atomic formulas of the form:

$$s_*(\varphi(w_1, \cdots, w_n)) = \varphi(sw_1, \cdots, sw_n).$$

The system of all such relations is denoted by $(*)$.

Now, by the definition, the algebra of compressed formulas $\mathrm{Hal}_{\Phi\Theta} = \mathrm{Hal}_\Theta(\Phi)$ is the result of factorization of the free Halmos algebra $\tilde{H}$ by the defining relations $(*)$. Its elements are called *the compressed formulas.*

Simultaneously, we defined also the Halmos category $\mathrm{Hal}_\Theta(\Phi)$.

The algebra $\mathrm{Hal}_\Theta(\Phi)$ is no more a free Halmos algebra, but it preserves some feature of freedom.

Let us consider (multi-sorted) mappings $\tau : M \to Q$, where Q is an arbitrary Halmos algebra in Θ and M is the set of atomic formulas. For every $X \in \Gamma^0$ we have

$$\tau_X : M_X \to Q(X).$$

We call a mapping τ *correct* if for every $s : W(X) \to W(Y)$:

$$\tau(sw \equiv sw') = s_*(\tau(w \equiv w')),$$

$$\tau(\varphi(sw_1, \cdots, sw_n)) = s_*\tau\varphi(w_1, \cdots, w_n),$$

for $w, w', w_1, \cdots, w_n \in W(X)$.

Theorem 2. *A mapping $\tau : M \to Q$ is extended up to a homomorphism τ : $\mathrm{Hal}_\Theta(\Phi) \to Q$ iff this mapping is correct.*

Proof. Take a homomorphism $\tilde{\tau} : \tilde{H} \to Q$ for the given $\tau : M \to Q$. Besides, take the natural homomorphism $\tilde{H} \to \mathrm{Hal}_\Theta(\Phi)$. The problem is to construct the homomorphism $\tau' : \mathrm{Hal}_\Theta(\Phi) \to Q$ with the commutative diagrams

$$\begin{tikzcd} M \arrow[r, "\tau"] \arrow[dr] & Q \\ & \mathrm{Hal}_\Theta(\Phi) \arrow[u, "\tau'"'] \end{tikzcd} \qquad \begin{tikzcd} \tilde{H} \arrow[r, "\tilde{\tau}"] \arrow[dr] & Q \\ & \mathrm{Hal}_\Theta(\Phi) \arrow[u, "\tau'"'] \end{tikzcd}$$

Such τ' exists if and only if the relations $(*)$ belong to $\mathrm{Ker}\tau$. But this exactly means that the mapping $\tau : M \to Q$ is correct.

In the sequel we will relate the condition of correctness of the mapping τ to the idea of the support of the elements of Halmos algebra.

3.2 The value of a formula.

We fix a model $(G, \Phi, f), G \in \Theta$, where f is the interpretation of the set Φ in G. For every n-ary relation $\varphi \in \Phi$ we have $f(\varphi) \subset G^{(n)}$. For the given model f we define the canonical mapping Val_f, which is a homomorphism of Halmos algebras

$$\mathrm{Val}_f : \mathrm{Hal}_\Theta(\Phi) \to \mathrm{Hal}_\Theta(G).$$

First we define it on atomic formulas. Take some $X \in \Gamma^0$ and consider formulas in M_X. Now define

$$Val_f(w \equiv w') = \mathrm{Val}(w \equiv w')$$

to be the equality in $\mathrm{Hal}_\Theta(G)(X)$. Thus,

$$Val_f(w \equiv w') = \{\mu : W(X) \to G | w^\mu = w'^\mu\}.$$

Analogously,

$$Val_f(\varphi(w_1, \cdots, w_n)) = \{\mu | (w_1^\mu, \cdots, w_n^\mu) \in f(\varphi)\}.$$

In particular, it is clear, that the set $f(\varphi)$ one-to-one corresponds to the set $\mathrm{Val}_f(\varphi(x_1, \cdots, x_n))$, all $x_1, \cdots, x_n$ are different, $X = \{x_1, \cdots, x_n\}$.

We have a multi-sorted mapping

$$\mathrm{Val}_f : M \to \mathrm{Hal}_\Theta(G).$$

The Halmos algebra $\tilde{H}$ is freely generated by the set M. Thus, we have a homomorphism of Halmos algebras

$$\mathrm{Val}_f : \tilde{H} \to \mathrm{Hal}_\Theta(G)$$

Simultaneously, we have a homomorphism of $\tilde{L}$-algebras

$$\mathrm{Val}_f : H \to \mathrm{Hal}_\Theta(G).$$

It is easy to check that the relations $(*)$ hold in every algebra $\mathrm{Hal}_\Theta(G)$, which means that they belong to the kernels of the homomorphisms above. This gives the homomorphism

$$\mathrm{Val}_f : \mathrm{Hal}_\Theta(\Phi) \to \mathrm{Hal}_\Theta(G)$$

that we do need.

For every formula u, pure or compressed, we have defined its value on the model (G, Φ, f) as $\mathrm{Val}_f(u)$. It is a subset $A = \mathrm{Val}_f(u)$ in the space $\mathrm{Hom}(W(X), G)$.

The kernel $\mathrm{Ker}(\mathrm{Val}_f)$ which lies in $\mathrm{Hal}_\Theta(\Phi)$ is a filter which may be considered as the *elementary theory of the model* (G, Φ, f).

Two formulas u and v of the same sort X in the algebra of pure formulas H are called *semantically equivalent* if for every model (G, Φ, f) the equality $\mathrm{Val}_f(u) = \mathrm{Val}_f(v)$ holds.

Theorem 3. *The formulas u and v are semantically equivalent iff they coincide in the algebra* $\mathrm{Hal}_\Theta(\Phi)$.

Proof. The proof will be given in 4.3.

Now let us introduce the *logical kernel* of a homomorphism $\mu : W(X) \to G$ by the rule $\mathrm{LogKer}(\mu) = \{u \in \mathrm{Hal}_{\Phi\Theta}(X) | \mu \in \mathrm{Val}_f(u)\}$. It is a Boolean filter, moreover, an ultrafilter in the algebra $\mathrm{Hal}_{\Phi\Theta}(X)$. The usual kernel $\mathrm{Ker}(\mu)$ can be considered as the set of all equalities $w \equiv w'$ in $\mathrm{LogKer}(\mu)$.

Every equality $w \equiv w'$ can be viewed as an equation in the algebra G. The set of all solutions of such equation is the set $\mathrm{Val}_f(w \equiv w') = \mathrm{Val}(w \equiv w')$. Now an arbitrary formula $u \in H$ can be also considered as an equation, but in the model (G, Φ, f). The set of all solutions of such "equation" u is the set $\mathrm{Val}_f(u)$.

Note here also that the point $\mu : W \to G$ is a solution of the equation $w \equiv w'$ if and only if $(w \equiv w') \in \mathrm{Ker}(\mu)$. The same μ is a solution of the "equation" u if and only if $u \in \mathrm{LogKer}(\mu)$. We use this point of view in the universal algebraic geometry.

4. Structure of the algebra $\mathrm{Hal}_\Theta(\Phi)$.

4.1 Elementary formulas.

The operation $s \in S_\Theta$ can be included in the formulas $u \in \mathrm{Hal}_{\Phi\Theta}(X)$. Formulas without such s are considered as elementary formulas. We prove here that every u of the a type X is equivalent in some natural sense to an elementary formula v of the type Y, $X \subset Y$.

The notion of formulas equivalence generalizes the notion of semantical equivalence. The corresponding formulas do not need to be of the same sort.

Let the set X be a subset of the set Y. The identical inclusion $X \to Y$ defines the identical homomorphism $s^0 : W(X) \to W(Y)$, with the corresponding

$$s^0 : H(X) \to H(Y),$$

$$s^0_* : \mathrm{Hal}_{\Phi\Theta}(X) \to \mathrm{Hal}_{\Phi\Theta}(Y).$$

Denote by $\overline{u}$ a compressed formula in $\mathrm{Hal}_{\Phi\Theta}(X)$, corresponding to a formula $u \in H$ of the type X. Let now u and v be two formulas of the types X_1 and X_2, respectively. We call these formulas equivalent if for some Y, containing X_1 and X_2, the equality

$$\overline{s^0_1 u} = \overline{s^0_2 v}$$

which gives

$$s_1^0\overline{u} = s_2^0\overline{v}$$

, holds. We call a formula $u \in H(X)$ an *elementary* one if it can be expressed by atomic formulas of the set M_X in the terms of the signature L_X.

The following theorem is the theorem on the elimination of the operations of the type $s \in S_\Theta$.

Theorem 4. *Every formula u in the algebra of pure formulas H is equivalent to some elementary formula.*

Proof. Denote by H^0 a subset in H, determined by the rule: $u \in H^0(X)$ if the formula u is equivalent to some elementary formula. It is clear that the set of all atomic formulas M is a part of the set H^0. We want to check that $H = H^0$. It is enough to verify that H^0 is a subalgebra in H.

Let us make some remarks on the notion of equivalence of two formulas.

Let v be an elementary formula of the type Y and the inclusion $s^0 : W(Y) \to W(Y')$ be given. Then the formula $s^0 v$ is equivalent to the elementary formula of the type Y'. Indeed, let us take a formula v' in $H(Y')$, whose record coincides with v, but in the set $M_{Y'}$. This v' is an elementary formula and it is easy to understand that $\overline{s_0 v} = \overline{v'}$. The last means that $s^0 v$ and v' are equivalent.

Let now u and v be two equivalent formulas of the types X_1 and X_2 respectively and Y a set, containing X_1 and X_2 with the inclusions

$$s_1^0 : W(X_1) \to W(Y), \;\; s_2^0 : W(X_2) \to W(Y),$$

such that $s_1^0\bar{u} = s_2^0\bar{v}$ holds true. Take now an arbitrary set Y', containing Y, with the inclusion $s^0 : W(Y) \to W(Y')$. Then we have also inclusions

$$s^0 s_1^0 : W(X)_1) \to W(Y'),$$
$$s^0 s_2^0 : W(X_2) \to W(Y'),$$

and

$$(s^0 s_1^0)\overline{u} = s^0(s_1^0\overline{u}) = s^0(s_2^0\overline{v}) = (s^0 s_2^0)\overline{v},$$
$$\overline{(s^0 s_1^0)u} \;=\; \overline{(s^0 s_2^0)v}.$$

Thus, we see that along with the set Y_1 containing X_1 and X_2 one can proceed from any Y' containing Y in the definition of the equivalence of two formulas.

Using these remarks it is easy to check that every set $H^0(X)$ is closed under the operations of the signature L_X. Now we need only to verify that the set H^0 is invariant under the operations of the type s.

Take $s : W(X) \to W(Y)$ and let u be a formula in $H^0(X)$. Check that $su \in H^0(Y)$.

The formula u is equivalent to an elementary formula. First, consider the situation when u is elementary itself. Apply induction by the record of the formula u in atomic formulas of the set M_X. Here X is a fixed set while Y and $s : W(X) \to W(Y)$ are arbitrary.

If u is an atomic formula, then the formula su is equivalent to the atomic formula which is an elementary one.

Let now u_1 and u_2 of the type X be elementary formulas and su_1 and su_2 be equivalent to the elementary formulas v_1 and v_2. Formulas su_1 and su_2 have the type Y. Let Y_1 be the type of formula v_1 and Y_2 the type of formula v_2. There is a set Y' containing Y, Y_1, Y_2 with the inclusions

$$s^0 : W(Y) \to W(Y'),$$

$$s_1^0 : W(Y_1) \to W(Y'),$$

$$s_2^0 : W(Y_2) \to W(Y'),$$

such that

$$\overline{(s^0 s)u_1} = \overline{s_1^0 v_1},$$

$$\overline{(s^0 s)u_2} = \overline{s_2^0 v_2}$$

hold.

Take $u = u_1 \vee u_2$. We have

$$\overline{su} = s\overline{u} = s(\overline{u}_1 \vee \overline{u}_2) = s\overline{u}_1 \vee s\overline{u_2};$$

$$(s^0 s)\overline{u} = s^0(s\overline{u}_1 \vee s\overline{u}_2) = (s^0 s)\overline{u}_1 \vee (s^0 s)\overline{u}_2 =$$

$$= \overline{s_1^0 v_1} \vee \overline{s_2^0 v_2} = \overline{s_1^0 v_1 \vee s_2^0 v_2}.$$

We use the first remark on the equivalence of formulas. We have:

$$\overline{(s^0 s)u} = \overline{s_1^0 v_1} \vee \overline{s_2^0 v_2} = \overline{v_1'} \vee \overline{v_2'} =$$

$$\overline{v_1' \vee v_2'} = \overline{v'};$$

The formula $v' = v_1' \vee v_2'$ is an elementary formula of the type Y', and $su \in H^0(Y)$.

Similarly one can check that if $u = u_1 \wedge u_2$ or $u = \neg u_1$, then $su \in H^0(Y)$.

Consider further the case $u = \exists x u_1$. As earlier, u and u_1 are of the type X and are elementary, and the assumption of the induction holds for u_1, i.e., su_1 is equivalent to an elementary formula for any s.

Let us turn to the formulas $su = s\exists x u_1$ and to $\overline{su} = s\overline{u} = s\exists x \overline{u}_1$.

The variable x belongs to the set X, and su belongs to the set $H(Y)$. Extend the set Y up to Y', adding an arbitrary variable y' to Y. Let $s^0 : W(Y) \to W(Y')$ be the corresponding identical inclusion. Pass to $s^0 s : W(X) \to W(Y')$ and consider a homomorphism $s_1 \colon W(X) \to W(Y')$ acting as $s^0 s$ on all elements from X except x, and $s_1 x = y'$. We have

$$(s^0 s)\overline{u} = (s^0 s)\exists x \overline{u_1} = s_1 \exists x \overline{u}_1.$$

The element $y' = s_1 x$ does not belong to the record of any $s_1 x_1, x_1 \neq x$ by construction. Then

$$(s^0 s)\overline{u} = \exists y' s_1 \overline{u}_1.$$

By the assumption of induction, the element $s_1 u_1$ is equivalent to the elementary v. Let v be of the type Y_1, $s_1 u_1$ has the type Y'. Take the set Y_1', containing Y_1 and Y', and let $s_1^0 \colon W(Y_1) \to W(Y_1')$ and $s_2^0 \colon W(Y') \to W(Y_1')$ be identical inclusions. Then $(s_2^0 s_1)\overline{u}_1 = s_1^0 \overline{v}$. We have also $(s_2^0 s^0 s)\overline{u} = s_2^0 s_1 \exists x \overline{u}_1 = s_2^0 \exists y' s_1 \overline{u}_1 = \exists y' s_2^0 s_1 \overline{u}_1 = \exists y' \overline{s_1^0 v} = \exists y' \overline{v^1}$, where v' is an elementary formula of the type Y_1'. The same is true for the formula $\exists y' v'$. We have:

$$\overline{(s_2^0 s^0)(su)} = \overline{\exists y' v'}, su \in H^0(Y).$$

This equality holds for every elementary formula u in $H^0(X)$.

Let us consider the case when u is not necessarily elementary. The formula u is equivalent to the elementary formula u', for example, of the type X_1. Take a set X' with the inclusion $s^0 : W(X) \to W(X')$ and s_1^0: $W(X_1) to W(X')$ and with the condition $\overline{s^0 u} = s^0 \overline{u} = s_1^0 \overline{u'} = \overline{s_1^0 u'}$. Since u' is an elementary formula, $\overline{s_1^0 u'} = \overline{v}$, where v is an elementary formula of the type X'. Thus, $\overline{s^0 u} = \overline{v}$.

Select a set Y' containing Y with the commutative diagram

$$\begin{tikzcd} W(X) \arrow[r, "s"] \arrow[d, "s^0"'] & W(Y) \arrow[d, "s_2^0"] \\ W(X') \arrow[r, "s_1"] & W(Y') \end{tikzcd}$$

where s^0, s_2^0 are inclusions, and $s_1 s^0 = s_2^0 s$. Let us apply it to $\overline{u}$:

$$s_1 s^0 \overline{u} = s_2^0 s \overline{u} = \overline{s_1 v}.$$

Here v is an elementary formula of the type Y', hence the formula $s_1 v$ is equivalent to the elementary formula v_1. Let v_1 be of the type Y_1 and let the set Y_1' contain Y_1 and Y'. We have the inclusions $s_3^0 \colon W(Y') \to W(Y_1')$, $s_4^0 \colon W(Y_1) \to W(Y_1')$ with the condition $s_3^0 s_1 \overline{v} = s_4^0 \overline{v_1} = \overline{s_4^0 v_1} = \overline{v'}$, where v' is an elementary formula of the type Y_1'. Now

$$s_3^0 s_1 \overline{v} = s_3^0 s_2^0 s \overline{u} = \overline{v'},$$

$$(s_3^0 s_2^0)(\overline{su}) = \overline{v'}.$$

Thus, the formula su is equivalent to the elementary formula, i.e., $su \in H^0(Y)$.

The theorem is proved.

4.2. Additional remarks.

Note first of all that if we specify the type X of the formula u, we simultaneously distinguish space where the value of the formula $\mathrm{Val}_f(u)$ lies. The formula $y^2 \equiv 2px$ determines a parabola in the two dimensional real space if $X = \{x, y\}$. The same formula for $X = \{x, y, z\}$ gives a cylinder in the three-dimensional space.

We view the formulas u and v as different ones if they are of different types. On the other hand, we may consider another equivalence of such formulas.

Let u and v be of the types X_1 and X_2, respectively. We consider them as equivalent ones if for some Y containing X_1, X_2 the equality $s_1^0 u = s_2^0 v$, where s_1^0 and s_2^0 are defined as above, holds. For $X_1 = X_2$ the formulas u and v are equivalent if they coincide.

Using this new equivalence we can link one-sorted and multi-sorted Halmos algebras. We will return to this later.

Proposition 1. *For every homomorphism* $\sigma : \mathrm{Hal}_\Theta(\Phi) \to \mathrm{Hal}_\Theta(G)$ *there exists a model* (G, Φ, f) *with the condition:* $\sigma = \mathrm{Val}_f$.

Proof. For every n-ary $\varphi \in \Phi$ we fix a set $X_\varphi = \{x_1, \cdots, x_n\}$ and consider an atomic formula $\varphi(x_1, \ldots, x_n)$ of the type $X = X_\varphi$. All these $\varphi(x_1, \ldots, x_n)$ and equality $x \equiv y$ of the type $\{x, y\}$ generate the whole algebra $H_\Theta \Phi$. We denote this set of generators by M^0.

Define

$$f(\varphi) = (\sigma\varphi(x_1, \dots, x_n))^{\pi}.$$

Here the bijection $\pi : \mathrm{Hom}(W(X), G) \to G^{(n)}$ if defined by

$$\pi(\mu) = (\mu(x_1), \dots, \mu(x_n)),$$

for every $\mu : W(X) \to G$.

This determines the model (G, Φ, f).

It is necessary to check that $\sigma = \mathrm{Val}_f$. We have:

$$\mathrm{Val}_f(\varphi(x_1, \dots, x_n)) = f(\varphi)^{\pi^{-1}} = (\sigma\varphi(x_1, \dots, x_n))^{\pi\pi^{-1}} = \sigma\varphi(x_1, \dots, x_n).$$

We used here the definition of the function Val_f on atomic formulas. As for equalities, note that by the definition of a homomorphism, every σ has to be correlated with equalities. This means, in particular, that $\sigma(x \equiv y)$ has to be the equality in $\mathrm{Hom}(W(x, y), G)$. By the definition we have $\sigma(x \equiv y) = \mathrm{Val}_f(x \equiv y)$. Thus, the equality $\sigma = \mathrm{Val}_f$ holds on the generators and, hence, it is always true.

■

Consider in more details the kernels of homomorphisms and filters in multi-sorted Halmos algebras.

Let $\tau : H \to H'$ be a homomorphism of multi-sorted Halmos algebras. For every $X \in \Gamma^0$ there is a homomorphism of quantorian X-algebras with equalities

$$\tau_X = \tau(X) : H(X) \to H'(X).$$

Its kernel $U(X) = \mathrm{Ker}(\tau_X)$ is a full coimage of the unit. It is a Boolean filter in $H(X)$, compatible with quantifiers. The condition of compatibility with quantifiers means that $\forall x u \in U(X)$ holds for every $x \in X$ and every $u \in U(X)$.

Here, as usual, $\forall u = \neg(\exists x \neg u)$. We say that the filter $U(X)$ is invariant in respect to universal quantifiers $\forall x, x \in X$.

Note also the conditions of compatibility with the operations of the type $s \in S_\Theta$: for every $s : W(X) \to W(Y)$ and every $u \in U(X)$ we have $su \in U(Y)$.

The kernel $\mathrm{Ker}(\tau) = U$ is a set of all $U(X) = \mathrm{Ker}(\tau_X)$ for all $X \in \Gamma^0$. All these $U(X)$ are compatible with universal quantifiers and operations of the type s. We defined a filter of a multi-sorted Halmos algebra. Straightforward check shows that

such filters and homomorphisms of Halmos algebras are tied correctly. Also ideals of multi-sorted Halmos algebras are defined.

The filter U is called *trivial*, if it always consists only of the unit of the algebra H. If $U(X) = H(X)$ holds for every X, then U is called an *unproper* filter. Such filters are always present in every H. If there are no other filters in H, then the algebra H is called *a simple* one.

Repeating arguments from [Pl1], one can prove that every algebra $\mathrm{Hal}_\Theta(G)$ and all its subalgebras are simple Halmos algebras and these are the only simple Halmos algebras (see 4.3).

Besides, we prove (see Lemma 2) that every Halmos algebra is semisimple: it is approximated by simple ones. The proof of Theorem 1 (see 4.3) uses all above.

Let us consider the notion of the support of an element of Halmos algebra. Let H be a Halmos algebra and u its element of the X type. Its *support* Δu is defined by the rule:

$$\Delta u = \Delta_X u = \{x \in X | \exists x u \neq u\}.$$

This condition means that the given x essentially participates in the "record" of the element u. However, speaking about the record of the element, one should take care. For example, an element $su = v$ for $s : W(X) \to W(Y)$ and $u \in H(X)$, has the type Y and its support is calculated in Y. Then, everything that is included in the record of u in $H(X)$ does not participate in the record of the element v.

Thus, the variables which do not belong to the set X may participate in the record of the element of the given type X, if operations of the type s are included in this record. These observations are formalized in the algebra of formulas $\mathrm{Hal}_\Theta(\Phi)$. For the elementary formulas the support of the formula and the variables participating in the record are perfectly coordinated. As a matter of fact, the notion of the support of an element in an arbitrary multi-sorted Halmos algebra requires a special consideration. For example, what can be said about the relations between the supports $\Delta_Y(su)$ and $\Delta_X(u)$ if u is of X type and $s : W(X) \to W(Y)$ is given? We may suppose that such relations take into account transitions from the variables $x \in X$ to the support of the elements $sx \in W(Y)$ (see also [Pl1]).

Proposition 2. *If $\tau : H \to H'$ is a homomorphism of Halmos algebras, then $\Delta_X(\tau u) \subset \Delta_X(u)$ for every $u \in H$ of the type X.*

Proof. Let $x \in X$ and $\exists x\tau(u) = \tau(\exists xu) \neq \tau u$. Then $\exists xu \neq u$, $x \in \Delta_X(u)$.

We refer to such feature as a coordination between homomorphisms and supports of the elements.

We have defined the set of generators M^0 earlier. This set can be viewed as a multi-sorted one. The supports of the formulas from M^0 are $\Delta\varphi(x_1, \cdots, x_n) = X_\varphi = \{x_1, \ldots, x_n\}$, $\Delta(x \equiv y) = \{x, y\}$. Let H' be a an arbitrary Halmos algebra. A multi-sorted mapping $\tau : M^0 \to H'$ is called coordinated with the supports of the elements, if for every $\varphi \in \Phi$ and every $X \in \Gamma^0$ with $X_\varphi \cup \{x, y\} \subset X$ the formulas

$$\Delta_X(\tau_\varphi(x_1, \ldots, x_n)) \subset X_\varphi \quad \text{and} \quad \Delta_X\tau(x \equiv y))) \subset \{x, y\}.$$

hold. Every such τ is uniquely extended up to $\tau : M \to H'$.

Take $w, w', w_1, \ldots, w_n$ of the type Y and consider $s : W(X) \to W(Y)$ with $s(x) = w, s(y) = w'$, $s(x_1) = w_1, \ldots, s(x_n) = w_n$. We set

$$\tau(\varphi(w_1, \ldots, w_n)) = s\tau(\varphi(x_1, \ldots, x_n)),$$
$$\tau(w \equiv w') = s\tau(x \equiv y).$$

Check that this definition does not depend on the choice of s.

Consider a set of variables $X_0 = \{x, y, x_1, \ldots, x_n\} \subset X$ and let s' coincide with s on the set X_0. Take $z \in X \setminus X_0$. We have:

$$s\tau(\varphi(x_1, \ldots, x_n)) = s\exists z\tau\varphi(x_1, \ldots, x_n) =$$
$$= s'\exists z\tau\varphi(x_1, \ldots, x_n) = s'\tau\varphi(x_1, \ldots, x_n).$$

Here $\tau\varphi(x_1, \ldots, x_n) = \exists z\tau\varphi(x_1, \ldots, x_n)$ by the definition of τ and z is not in the support for $\tau\varphi(x_1, \ldots, x_n)$. The transition from s to s' is fulfilled according to the rules of Halmos algebra. Similarly, $s\tau(x \equiv y) = s'\tau(x \equiv y)$. Thus, $\tau : M \to H'$. We will check that the mapping τ is a correct one. Hence, it is uniquely extended up to the homomorphism $\tau : \mathrm{Hal}_\Theta(\Phi) \to H'$.

Take some $s : W(X) \to W(Y)$ and let $w, w', w_1, \ldots, w_n$ be of the type X. Take also a set X_0 containing variables $x, y, x_1, \ldots, x_n$ and let $s_0 : W(X_0) \to W(X)$ be defined as s above. Then $\tau\varphi(sw_1, \ldots, sw_n) = \tau\varphi(ss_0x_1, \ldots, ss_0x_n) = ss_0\tau\varphi(x_1, \ldots, x_n) = s\tau\varphi(s_0x_1, \ldots, s_0x_n) = s\tau\varphi(w_, \ldots, w_n)$. Similarly, $\tau(sw \equiv sw') = s\tau(w \equiv w')$. Hence, the mapping τ is correct. This property has been mentioned in Theorem 2. The following Proposition is also related to the important properties of the algebra $\mathrm{Hal}_\Theta(\Phi)$.

Proposition 3. *Let $\alpha : H \to H'$ be a surjective homomorphism of two (multi-sorted) Halmos algebras. Let a homomorphism $\beta : Hal_{\Phi\Theta} \to H'$ be given. Then, there exists : $\gamma : Hal_{\Phi\Theta} \to H$ such that $\alpha\gamma = \beta$.*

Proof. Let us use, first, an auxiliary remark. Let h be an element in H of the type X. Then $\Delta_X(\alpha(h)) \subset \Delta_X(h)$. Show that there exists an element h_1 in H of the type X, such that $\alpha(h) = \alpha(h_1)$ and $\Delta_X(h_1) = \Delta_X\alpha(h_1)$.

Let us take

$$Y = \Delta_X(h)\backslash\Delta_X(\alpha(h)),$$

and let $h_1 = \exists(Y)h$. Since Y does not intersect the support of the element $\alpha(h)$ then

$$\alpha(h_1) = \alpha(\exists Y h) = \exists(Y)\alpha(h) = \alpha(h).$$

Besides that

$$\Delta_X(h_1) = \Delta_X(h)\backslash Y = \Delta_X\alpha(h) = \Delta_X\alpha(h_1),$$

and $\Delta_X(h_1) = \Delta_X\alpha(h_1)$.

Now we use this remark. For every basic element u of the type X in $\mathrm{Hal}_{\Phi\Theta}$ we take $\beta(u)$. This is an element of the type X in H'. Take an element $h \in H$ with $\alpha(h) = \beta(u)$. Let us take h_1 such that $\alpha(h_1) = \alpha(h) = \beta(u)$ and

$$\Delta_X(h_1) = \Delta_X\alpha(h_1) = \Delta_X\beta(u) \subset \Delta_X(u).$$

Denote $\gamma(u) = h_1$. Thus, for every u we have:

$$\Delta_X(\gamma(u) = \Delta_X(h_1) \subset \Delta_X(u).$$

This means that the homomorphism $\gamma : \mathrm{Hal}_{\Phi\Theta} \to H$ is defined.

For every basic u we have $(\alpha\gamma)(u) = \alpha(h_1) = \alpha(h) = \beta(u)$. Then $(\alpha\gamma)(u) = \beta(u)$ for every u, i.e., $\alpha\gamma = \beta$.

Now we pass to the proof of Theorem 3.

Proof. This commutative diagram follows from the definitions:

$$\begin{tikzcd} H \arrow[rr, "\mathrm{Val}_f"] \arrow[dr] & & \mathrm{Hal}_\Theta(G) \\ & \mathrm{Hal}_\Theta(\Phi) \arrow[ur, "\mathrm{Val}_f"'] & \end{tikzcd}$$

This means that $\mathrm{Val}_f(u) = \mathrm{Val}_f(\overline{u})$ holds for every formula $u \in H$. If, now, $\overline{u} = \overline{v}$, then $\mathrm{Val}_f(u) = \mathrm{Val}_f(\overline{u}) = \mathrm{Val}_v(\overline{v}) = \mathrm{Val}_f(v)$. It is true for every model (G, Φ, f). Thus, $\overline{u} = \overline{v}$ implies semantical equivalence of the formulas u and v.

Conversely, let u and v be semantically equivalent. Apply the fact that every Halmos algebra, is semisimple. In particular, the algebra $\mathrm{Hal}_\Theta(\Phi)$ is semisimple. We have a system of homomorphisms $\sigma_\alpha : \mathrm{Hal}_\Theta(\Phi) \to H_\Theta(G_\alpha), \alpha \in I$, such that if $\sigma_\alpha(u) = \sigma_\alpha(v)$ for all α, then $u = v$. Every σ_α here can be represented as Val_{f_α} by a model $(G_\alpha, \Phi, f_\alpha)$. If u and v are semantically equivalent formulas in $\mathrm{Hal}_\Theta(\Phi)$, then $\mathrm{Val}_{f\alpha}(u) = \mathrm{Val}_{f\alpha}(v), \sigma_\alpha(u) = \sigma_\alpha(v)$ for all $\alpha \in I$. This gives $u = v$.

Note that similar reasoning does not work, for example, for the algebra $\tilde{H}$. Here $\sigma : \tilde{H} \to \mathrm{Hal}_\Theta(G)$ can be represented as $\sigma = \mathrm{Val}_f$ only if σ is coordinated with the relations $(*)$.

4.3 Some proofs.

Here we give proof of the theorem 1.

Besides filters, consider ideals. An ideal U of the Halmos algebra H selects in $H(X)$ a Boolean ideal $U(X)$ for every $X \in \Gamma^0$, which is invariant under quantifiers $\exists x, x \in X$, and for every $s : W(X) \to W(Y)$ we have: $u \in U(X)$ implies $su \in U(Y)$. Ideals and filters are dual. The zero ideal is an ideal U for which all $U(X)$ are zeroes. The ideal U is trivial, if $U(X) = H(X)$ for all X. The algebra H is simple if and only if it has no other ideals.

Lemma 1. *Every* $\mathrm{Hal}_\Theta(G)$ *and all its subalgebras are simple.*

Proof. Take a nonempty subset A in $\mathrm{Hom}(W(X), G)$. Apply a quantifier $\exists(X) = \exists x_1 \ldots \exists x_n, X = \{x_1, \ldots, x_n\}$. The homomorphism $\mu : W(X) \to G$ belongs to the set $\exists(X)A$ if μ coincides with some $\nu : W(X) \to G$ outside X. Since there are no variables outside X, then every μ belongs to $\exists(X)A$; $\exists X A = \mathrm{Hom}(W(X), G)$. It is the unit of the algebra $\mathrm{Bool}(W(X), G)$.

Let now H be an arbitrary Halmos algebra such that $\exists X a = 1$ holds for every nonzero element $a \in H(X)$ for every $X \in \Gamma^0$.

Check that H is simple. Let U be a nonzero ideal in H. For some X we have $U(X) \neq 0$ and in $U(X)$ there is a nonzero element a. We have $\exists x a = 1$ that is $U(X)$ contains a unit. Then $U(X) = H(X)$. For any $s : W(X) \to W(Y)$ we have $su \in U(Y)$ for $u \in U(X)$. If u is a unit in $H(X)$, then su is a unit in $H(Y)$, $U(Y)$

contains a unit, and $U(Y) = H(Y)$. The ideal U is trivial, that is the algebra $\mathrm{Hal}_\Theta(G)$ and all its subalgebras are simple.

We will prove later that every simple Halmos algebra is a subalgebra of some $\mathrm{Hal}_\Theta(G)$.

We use the following definition.

Let H be a multi-sorted Halmos algebra. We say that this algebra is *one-sorted representable* if there exists a one-sorted Halmos algebra $H(X^0) = H^0$ such that

1. To each inclusion $s^0 : W(X) \to W(X^0)$ there corresponds a Boolean inclusion $s^0 : H(X) \to H(X^0)$, naturally correlated with quantifiers and equalities.

2. To each commutative diagram

$$\begin{tikzcd} W(X) \arrow[r, "s"] \arrow[d, "s^0_X"'] & W(Y) \arrow[d, "s^0_Y"] \\ W(X^0) \arrow[r, "s'"] & W(X^0) \end{tikzcd}$$

there corresponds a diagram

$$\begin{tikzcd} H(X) \arrow[r, "s"] \arrow[d, "s^0_X"'] & H(Y) \arrow[d, "s^0_Y"] \\ H^0 \arrow[r, "s'"] & H^0 \end{tikzcd}$$

As in [Pl1], we can prove that every H is representable. It is easy to check directly for the algebras $\mathrm{Hal}_\Theta(\Phi)$ and $\mathrm{Hal}_\Theta(G)$. This notion gives ties between multi-sorted and one-sorted Halmos algebras. We use this fact of representability in the following theorem.

Theorem 5. *Every Halmos algebra is semisimple.*

Proof. Let the representation $H \to H^0$ be given. For every element $a \in H^0$ there is a maximal Boolean ideal V not containing the element a. Let V_* be the ideal of Halmos algebra H^0 defined by the rule: an element $u \in H^0$ belongs to V_* if $\exists(X^0)u \in V$. We may think that the algebra H^0 is locally finite. Then, $\exists X^0 u = \exists(X)u$, where $X \in \Gamma^0$. It is proved that V_* is a maximal ideal of the Halmos algebra H^0 and $V_* \subset V$ (see [Pl 1]).

Relying on V_* we build an ideal U in the multi-sorted algebra H. For every finite $X \subset X^0$ denote by $U(X)$ a set of elements $u \in H(X)$ for which $s^0 u \in V_*$. Here, $s^0 : H(X) \to H(X^0) = H^0$ is the natural embedding.

It is easy to check that $U(X)$ is a Boolean ideal in $H(X)$, preserving the quantifier $\exists(X)$. Check also that if $u \in U(X)$ then $su \in U(Y)$ for every for every $s : W(X) \to W(Y)$. Consider a commutative diagram

$$\begin{array}{ccc} H(X) & \xrightarrow{s} & H(Y) \\ {\scriptstyle s_X^0}\downarrow & & \downarrow{\scriptstyle s_Y^0} \\ H^0 & \xrightarrow{s'} & H^0 \end{array}$$

We have: $s_Y^0 s = s' s_X^0$. We need $s_Y^0(su) \in V_*$. Now $s_Y^0 su = s' s_X^0 u$. By the condition, $s_X^0 u \in V_*$. Then, $s' s_X^0 u \in V_*$, $s_Y^0(su) \in V_*$. Hence, U is an ideal of the algebra H. Let us show that U is a maximal ideal. Let U_1 be a greater one. An ideal U_* in H^0, determined by U_1, corresponds to it, and $U_* > V_*$. Since V_* is maximal, then $U_* = V_*$. It should be $U_1(X) > U(X)$ for some finite X. Let $u \in U_1(X) \setminus U(X)$. We have $s^0 u \in U_* = V_*$, and then $u \in U(X)$.

The arising contradiction means that the ideal U is maximal.

Take, further, an element $a \in H(X)$ and $s^0 a \in H^0$. The element $s^0 a$ determines the maximal Boolean ideal V and, correspondingly, the ideal V_* of the Halmos algebra. Let U be the filter of the Halmos algebra H corresponding to V_*. Denote $U = U_a$. Take these U_a for all nonzero elements $a \in H$ of the different types X.

All U_a are maximal ideals in H and their intersection is zero ideal. This means that the algebra H is semisimple, since it is approximated by simple algebras.

Lemma 2. *Every simple algebra H is isomorphic to some subalgebra of the algebra of the type* $\mathrm{Hal}_\Theta(G)$.

Proof. Let H be simple and H_0 some set of generating elements in H. For every $h \in H_0$ take a symbol of relation φ. If h is of the type $X = \{x_1, \dots, x_n\}$ then φ is n-ary. Associate $\varphi(x_1, \dots, x_n) \to h$. Recall that we consider every H as an algebra with equalities. Then the formula $w \equiv w'$ is associated with the corresponding equality in H.

Collect all φ into a set Φ and take the corresponding free algebra $\tilde{H}$ for Φ and Θ. This gives a homomorphism of the algebra $\tilde{H}$ on the algebra H. Let the ideal U be a kernel of such homomorphism. We use here one-sorted representation for the algebras $\tilde{H}$ and H, and let an ideal U_0 correspond to the ideal U in the algebra $\tilde{H}^0$. The algebra $\tilde{H}^0$ is a free one-sorted Halmos algebra. Take a filter T_0 corresponding to the ideal U_0. Since the algebra H is simple, the filter T_0 is a maximal one and

U is a maximal ideal. Filter T in the algebra H, associated with the ideal U, corresponds to the filter T_0.

The filter T_0 determines some model (G, Φ, f), in which the set of formulas T_0 and also all the formulas of the filter T hold. Take $\mathrm{Val}_f\colon \tilde{H} \to \mathrm{Hal}_\Theta(G)$ by f. Here, $T = \mathrm{Ker}(\mathrm{Val}_f)$. From this follows that the algebra H is isomorphic to some subalgebra in $\mathrm{Hal}_\Theta(G)$.

Lemmas 1 and 2 together with Theorem 5 imply Theorem 2 with the help of characterization of the varieties by Birkhoff type theorem for Halmos algebras.

5. Algebraic geometry in first order logic

5.1 Sets of formulas and algebraic sets.

Fix a finite set X. Let $W = W(X)$ be the free algebra over X in the given variety Θ. The set Φ of symbols of relations and a model $(G, \Phi, f), G \in \Theta$ are also fixed. We view the set of homomorphisms $\mathrm{Hom}(W, G)$ as an affine space (see also 1.3) .

Now, consider the sets A of points $\mu\colon W \to G$ in this space and the sets T of formulas in the algebra $\mathrm{Hal}_{\Phi\Theta}(X)$. Establish the following Galois correspondence for the given model (G, Φ, f):

$$\begin{cases} T^f = A = \bigcap\limits_{u \in T} Val_f(u) = \{\mu \big| T \subset \mathrm{LogKer}(\mu)\} \\ A^f = T = \{u \big| A \subset Val_f(u)\} = \bigcap\limits_{\mu \in A} \mathrm{LogKer}(\mu) \end{cases}$$

In each row we have three equalities, first two of which are the definition, while the third one is an easily checked equality. These equalities actually hold and this is a Galois correspondence. It generalizes the standard correspondence in the classical algebraic geometry and, also, the Galois correspondence for the equational geometry in the variety Θ for the algebra $G \in \Theta$.

The set A of the type $A = T^f$ for some T we call an *algebraic set, or closed set (algebraic variety)*, over the model (G, Φ, f). It is also called an elementary set.

The set T of the type $T = A^f$ for some A is a Boolean filter in the Boolean algebra $\mathrm{Hal}_{\Phi\Theta}(X)$. We call it an *$f$-closed Boolean filter*. If A is an algebraic set, then we consider the filter $T = A^f$ as a theory of the set A in the algebra $\mathrm{Hal}_{\Phi\Theta}(X)$.

Now we consider the Boolean algebra $\mathrm{Hal}_{\Phi\Theta}(X)/A^f$. It is the coordinate algebra

for the given A. We have the following embedding:

$$\mathrm{Hal}_{\Phi\Theta}(X)/A^f \to \prod_{\mu \in A} \mathrm{Hal}_{\Phi\Theta}(X)/\mathrm{LogKer}(\mu).$$

The right side is the cartesian product of two-element algebras.

We can consider the closure A^{ff} for every set A and T^{ff} for every T. The next proposition gives a straightforward relation between T and T^{ff}.

Proposition 4. *$v \in T^{ff}$ if and only if the (infinitary) formula $(\bigwedge_{u \in T} u) \to v$ holds in the model (G, Φ, f).*

Proof. The given formula holds in (G, Φ, f) if and only if for arbitrary point $\mu : W \to G$ the inclusion $\mu \in \mathrm{Val}_f(u)$ follows from that of $\mu \in \mathrm{Val}_f(u)$ for all $u \in T$. Let the formula hold in the model. Prove that $v \in T^{ff}$. By definition,

$$T^{ff} = \bigcap_{\mu \in T^f} \mathrm{LogKer}(\mu).$$

We need to prove that if $\mu \in T^f$, then $v \in \mathrm{LogKer}(\mu)$. We have

$$T^f = \bigcap_{u \in T} \mathrm{Val}_f(u).$$

Hence, $\mu \in T^f$ gives $\mu \in \mathrm{Val}_f(u)$ for all $u \in T$. Besides, $\mu \in \mathrm{Val}_f(v), v \in \mathrm{LogKer}(\mu)$ by the condition of the proposition. Thus, $v \in T^{ff}$.

Let now $v \in T^{ff}$. Prove that the formula holds in (G, Φ, f).

Take some $\mu : W \to G$ and let $\mu \in \mathrm{Val}_f(u)$ for all $u \in T$. We need to prove that $\mu \in \mathrm{Val}_f(v)$. We have:

$$\mu \in \bigcap_{u \in T} Val_f(u) = T^f, \quad T^{ff} \subset \mathrm{LogKer}(\mu).$$

Here $v \in T^{ff}$ means that $v \in \mathrm{LogKer}(\mu), \mu \in Val_f(v)$.

The proposition is proved.

Corollary. *If u is a formula, then $v \in u^{ff}$ if and only if the formula $u \to v$ holds in the model.*

This is a version of the Hilbert's Nullstellensats in the algebraic geometry.

5.2 The Galois correspondence and morphisms.

Further the Galois correspondence will be associated with the morphisms of the categories. In the case of $\mathrm{Hal}_\Theta(G)$ the homomorphisms $s\colon W(X) \to W(Y)$ yield the mappings s_* and s^* for Booleans. Now, let us define the action of s_* and s^* on the set of formulas T. If $T \subset \mathrm{Hal}_\Theta(\Phi)(Y)$, then s_*T is the set of formulas in $\mathrm{Hal}_\Theta(\Phi)(X)$, determined by the rule:

$$u \in s_*T \Leftrightarrow s_*u \in T.$$

If $T \subset \mathrm{Hal}_\Theta(\Phi)(X)$, then $s^*T \subset \mathrm{Hal}_\Theta(\Phi)(Y)$, and

$$s^*T = \{s_*u | u \in T\}.$$

Theorem 6. *The rules of compatibility are of the form:*

1. If $T \subset \mathrm{Hal}_\Theta(\Phi)(X)$, then

$$(s^*T)^f = s_*T^f = sT^f.$$

2. If $A \subset \mathrm{Hom}(W(Y), G)$, then

$$(s^*A)^f = s_*A^f.$$

Proof. We use the definitions:

$$T^f = \bigcap_{u\in T} \mathrm{Val}_f(u); \quad A^f = \{u \big| A \subset \mathrm{Val}_f(u)\}.$$

Let us prove the first rule. Due to $s^*T \subset \mathrm{Hal}_\Theta(\Phi)(Y)$, we have $(s^*T)^f,\ s_*T^f \subset \mathrm{Hom}(W(Y), G)$. Consider the point $\nu : W(Y) \to G$. Let $\nu \in (s^*T)^f = \bigcap\limits_{u\in s^*T} \mathrm{Val}_f(u)$, $u = sv, v \in T$.

For every $v \in T$ we have:

$$\nu \in \mathrm{Val}_f(sv) = s\mathrm{Val}_f(v) = s_*\mathrm{Val}_f(v).$$

Hence, for every $v \in T$ we have $\nu s \in \mathrm{Val}_f(v)$, and $\nu s \in T^f = \bigcap\limits_{v\in T} \mathrm{Val}_f(v), \nu \in sT^f$.

Conversely, let $\nu \in sT^f, \nu s \in T^f = \bigcap_{v\in T} \mathrm{Val}_f(v)$. Then $\nu s \in \mathrm{Val}_f(v)$, $\nu \in s\mathrm{Val}_f(v) = \mathrm{Val}_f(sv)$ for every $v \in T$. Therefore

$$\nu \in \bigcap_{v\in T} \mathrm{Val}_f(sv) = \bigcap_{u\in s^*T} \mathrm{Val}_f(u) = (s^*T)^f.$$

Consider the second rule. Here, $s^*A \subset \mathrm{Hom}(W(X), G)$ and $(s^*A)^f$, $s_*A^f \subset \mathrm{Hal}_\Theta(\Phi)(X)$. Let $u \in (s^*A)^f$, which implies $s^*A \subset \mathrm{Val}_f(u)$. Hence, $\nu s \in \mathrm{Val}_f(u)$, $\nu \in s\mathrm{Val}_f(u) = \mathrm{Val}_f(su)$, $A \subset \mathrm{Val}_f(su), su \in A^f, u \in s_*A^f$ for every $\nu \in A$.

Conversely, let $u \in s_*A^f, su \in A^f, A \subset \mathrm{Val}_f(su) = s\mathrm{Val}_f(u)$. Then $\nu s \in \mathrm{Val}_f(u)$, $s^*A \subset \mathrm{Val}_f(u), u \in (s^*A)^f$ for every $\nu \in A$.

The rules are proved.

The given rules induce a good compatibility between the Galois correspondence in logic and the morphisms in the categories $\mathrm{Hal}_\Theta(\Phi)$ and $\mathrm{Hal}_\Theta(G)$.

5.3 Lattices and topology.

We introduce here the functions Cl_f and Alv_f, defined for an arbitrary $W = W(X)$.

The set $Alv_f(W)$ is the set of all algebraic sets for the given model (G, Φ, f) in the affine space $\mathrm{Hom}(W(X), G)$.

The set $Cl_f(W)$ is the set of all f-closed filters T in the Boolean algebra $\mathrm{Hal}_\Theta(\Phi)(X)$.

Proposition 5. *The sets $Alv_f(W)$ and $Cl_f(W)$ are dually isomorphic lattices.*

Proof. Let $A, B \in Alv_f(W)$, $A = T_1^f$, $B = T_2^f$. Then $A \cap B = (T_1 \cup T_2)^f, A \cap B \in Alv_f(W)$.

1. If $A = T^f$ is an algebraic set, then sA is also the algebraic set.
2. If $T = A^f$ is f-closed, then $sT = s_*T$ is also f-closed.

Denote the set of all $u \wedge v, u \in T_1, v \in T_2$ by $T_1 \wedge T_2$. Then $A \cup B = (T_1 \cap T_2)^f$, $A \cup B \in Alv_f(W)$. So, $Alv_f(W)$ is a lattice, which is a sublattice in the lattice $\mathrm{Bool}(W(X), G)$, which is a distributive lattice.

Let, now $T_1, T_2 \in Cl_f(W), T_1 = A^f$, $T_2 = B^f$. Then $T_1 \cap T_2 = A^f \cap B^f = (A \cup B)^f$, $T_1 \cap T_2 \in Cl_f(W)$.

However, we can not proclaim that $T_1 \cup T_2$ also belongs to $Cl_f(W)$, since the union of filters can be not a filter.

Thus, we introduce a new operation:

$$T_1 \overline{\cup} T_2 = (T_1 \cup T_2)^{ff}.$$

As a result, we have a lattice $Cl_f(W)$. Check, that the lattices $Alv_f(W)$ and $Cl_f(W)$ are dually isomorphic. This will imply that the lattice Cl_f is also distributive.

The transition f determines the bijection between algebraic sets in $\mathrm{Hom}(W,G)$ and f-closed filters in $\mathrm{Hal}_\Theta(\Phi)(X)$. In other words, we have bijections

$$f\colon Alv_f(W) \to Cl_f(W),$$
$$f\colon Cl_f(W) \to Alv_f(W).$$

Let $A, B \in Alv_f(W)$. Then

$$(A \cap B)^f = (T_1^f \cap T_2^f)^f = (T_1 \cup T_2)^{ff} = T_1 \overline{\cup} T_2,$$

where $T_1 = A^f, T_2 = B^f$. Thus,

$$(A \cap B)^f = A^f \overline{\cup} B^f.$$

Now, we need to prove

$$(A \cup B)^f = A^f \cap B^f.$$

Since the set $A \cup B$ is closed, we need to check that $(A^f \cap B^f)^f = A \cup B$. We have:

$$A^f \cap B^f = (A \cup B)^f, (A^f \cap B^f)^f = (A \cup B)^{ff} = A \cup B.$$

The proposition is proved.

We can consider the function $Cl_f\colon \Theta^0 \to Set$ as a functor. However, there is no compatibility with the lattice $Cl_f(W)$ in this case. On the other hand, the function $Alv_f\colon \Theta^0 \to Set$ is compatible with the lattice and one can look at the functor

$$Alv_f : \Theta^0 \to Lat,$$

where Lat is the category of lattices.

Consider the function Alv_f also as a multi-sorted set. We have $Alv_f(X) = Alv_f(W(X))$ for every $X \in \Gamma^0$. Now, Alv_f is a subset in the algebra $\mathrm{Hal}_\Theta(G)$ which can be treated as a multi-sorted lattice, invariant under the operations of $s \in S_\Theta$ type.

Let us return to the homomorphism $\mathrm{Val}_f\colon \mathrm{Hal}_\Theta(\Phi) \to \mathrm{Hal}_\Theta(G)$. Let R_f be the image of this homomorphism. It s a subalgebra in $\mathrm{Hal}_\Theta(G)$. Every $A \in R_f$ has the form $A = \mathrm{Val}_f(u)$. It is an algebraic set, determined by one-element set T, consisting of an element u. We call such algebraic sets simple ones.

All simple algebraic sets for the given model (G, Φ, f) form a subalgebra in the Halmos algebra $\mathrm{Hal}_\Theta(G)$.

As we know, every formula u is equivalent to some elementary formula v. In the sequel we will relate the varieties $\mathrm{Val}_f(u)$ and $\mathrm{Val}_f(v)$.

Note, further, that the main attention in groups and other structures in classical algebraic geometry and equational geometry is paid to the properties of individual algebraic sets. The most important goal is to obtain some description of the system of all solutions of equations.

We pay main attention here to lattices and categories of algebraic sets. The properties of individual algebraic sets is a separate topic.

Let us give some remarks on topologies in the spaces $\mathrm{Hom}(W, G)$, associated with algebraic sets, namely, generalized Zarisski topologies. Closed sets here are algebraic sets. It is natural, however, to restrict ourselves with the sets, determined by a collection of positive formulas, recorded without negation. The good topology assumes also that there are no quantifiers (it can be explained). We speak of algebraic sets, determined by the collections of universal positive formulas, and consider such Zariski topology.

Example. Proceed from the classical variety Θ of commutative and associative algebras with the unit over the field of real numbers. Take a unique relation (order relation) as Φ. The model (G, Φ, f) is a field of real numbers, in which the order relation is naturally realized. Take $X = \{x, y\}$. The corresponding algebra W is the algebra of real polynomials of two variables x and y. The space $\mathrm{Hom}(W, G)$ is realized as the real space.

It is easy to understand that in this case the corresponding generalized Zariski topology coincides with the natural topology, while the usual Zariski topology does not coincide with this natural one. Take, in particular, a disk $x^2 + y^2 \leq a^2$. It is closed in the natural topology, but not closed in the usual Zariski topology. The same disk can be given by the formula $\exists z(x^2 + y^2 + z^2 = a^2)$. The set Φ is empty here, but there is a quantifier and the set X consists of three variables.

5.4 The categories $K_{\Phi\Theta}(f)$ and $C_{\Phi\Theta}(f)$.

Besides the variety of algebras Θ, let us fix also the set of symbols of relations Φ and a model $(G, \Phi, f), G \in \Theta$. These data determine the category $K_{\Phi\Theta}(f)$ of

algebraic sets for the given variety f. Its objects have the form

$$(X, A),$$

where A is an algebraic set in the given logic for the given model. The set A lies in the affine space $\mathrm{Hom}(W(X), G)$, $A = T^f$ for some set of formulas T, but this T is not fixed.

The morphisms have the form

$$(X, A) \to (Y, B).$$

We start from $s\colon W(Y) \to W(X)$ in Θ^0. We have:

$$\tilde{s}\colon \mathrm{Hom}(W(X), G) \to \mathrm{Hom}(W(Y), G).$$

We say that s is admissible for A and B if $\tilde{s}(\nu) = \nu s \in B$ for $\nu \in A$. For every such s we have a mapping

$$[s]\colon A \to B.$$

Now we consider weak and exact categories $K_{\Phi\Theta}(f)$.

In the weak category morphisms are of the form

$$s\colon (X, A) \to (Y, B),$$

where s is admissible for A and B.

In the exact category morphisms are

$$[s]\colon (X, A) \to (Y, B).$$

Consider the notion of admissible triple (s, A, B) in more detail. For $s\colon W(Y) \to W(X)$ we have

$$s_*\colon \mathrm{Hal}_{\Phi\Theta}(Y) \to \mathrm{Hal}_{\Phi\Theta}(X).$$

Let T_2 and T_1 be some sets of formulas in $\mathrm{Hal}_{\Phi\Theta}(Y)$ and $\mathrm{Hal}_{\Phi\Theta}(X)$, respectively. We say that s_* is admissible for T_2 and T_1 if $s_* u \in T_1$ for $u \in T_2$.

Proposition 6. *The homomorphism $s\colon W(Y) \to W(X)$ is admissible for algebraic sets A and B if and only if s_* is admissible for $T_2 = B^f$ and $T_1 = A^f$.*

Proof. Note first, that the inclusion $\nu s \in B$ for all $\nu \in A$ means that $A \subset sB = s_*B$. This holds in the category $\mathrm{Hal}_\Theta(G)$. Applying the transition f, we get $(s_*B)^f \subset A^f$. We claim further: $s^*B^f \subset (s_*B)^f$. This rule was not among the previously mentioned ones.

Take $u \in s^*B^f$, $u = s_*v, v \in B^f$, $B \subset \mathrm{Val}_f(v)$. From this follows $s_*B \subset s_*\mathrm{Val}_f(v) = \mathrm{Val}_f(s_*v) = \mathrm{Val}_f(u)$. Hence, $s_*B \subset \mathrm{Val}_f(u), u \in (s_*B)^f$. The last inclusion holds for ever $u \in s^*B^f$. Thus, we have $s^*B^f \subset (s_*B)^f$.

Let now s be admissible for A and B. We have $A \subset sB$ and $(s_*B)^f \subset A^f = T_1$. Using $s^*B^f \subset (s_*B)^f$, we get $s^*B^f = s^*T_2 \subset T_1$. This means that $s_*u \in T_1$ for $u \in T_2$ and s_* is admissible for T_2 and T_1.

Conversely, let now $s^*T_2 \subset T_1, s^*B^f \subset A^f$. Again applying f, we get

$$T_1^f = A \subset (s^*T_2)^f = s_*T_2^f = s_*B = sB$$

and s is admissible for A and B. The proposition is proved.

In this case we have a commutative diagram of Boolean homomorphisms

$$\begin{array}{ccc} \mathrm{Hal}_{\Phi\Theta}(Y) & \xrightarrow{s_*} & \mathrm{Hal}_{\Phi\Theta}(X) \\ \downarrow{\scriptstyle \mu_Y} & & \downarrow{\scriptstyle \mu_X} \\ \mathrm{Hal}_{\Phi\Theta}(Y)/B^f & \xrightarrow{\overline{s_*}} & \mathrm{Hal}_{\Phi\Theta}(X)/A^f \end{array}$$

where μ_Y and μ_X are natural homomorphisms.

Proposition 7. *Let s_1 and s_2 be admissible for A and B. Then $\overline{s_{1*}} = \overline{s_{2*}}$ follows from $[s_1] = [s_2]$.*

Proof. Let $[s_1] = [s_2]$. Then $\nu s_1 = \nu s_2$ for every $\nu \in A$. Consider the following two diagrams:

$$\begin{array}{ccc} \mathrm{Hal}_{\Phi\Theta}(Y) & \xrightarrow{s_{1*}} & \mathrm{Hal}_{\Phi\Theta}(X) \\ \downarrow{\scriptstyle \mu_Y} & & \downarrow{\scriptstyle \mu_X} \\ \mathrm{Hal}_{\Phi\Theta}(Y)/B^f & \xrightarrow{\overline{s_{1*}}} & \mathrm{Hal}_{\Phi\Theta}(X)/A^f \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Hal}_{\Phi\Theta}(Y) & \xrightarrow{s_{2*}} & \mathrm{Hal}_{\Phi\Theta}(X) \\ \downarrow{\scriptstyle \mu_Y} & & \downarrow{\scriptstyle \mu_X} \\ \mathrm{Hal}_{\Phi\Theta}(Y)/B^f & \xrightarrow{\overline{s_{2*}}} & \mathrm{Hal}_{\Phi\Theta}(X)/A^f \end{array}$$

We claim $\overline{s_{1*}} = \overline{s_2*}$, that is, $\overline{s_{1*}}\mu_Y(u) = \overline{s_{2*}}\mu_Y(u)$ for all $u \in \mathrm{Hal}_{\Phi\Theta}(Y)$. Hence, $\mu_X s_{1*}(u) = \mu_X s_{2*}(u)$.

The idea is to prove that $s_{1*}(u)$ and $s_{2*}(u)$ coincide modulo the filter A^f. This means that $(\neg s_{1*}(u) \vee s_{2*}(u)) \wedge (s_{1*}(u) \vee \neg s_{2*}(u)) \in A^f$. Denote the left part by v. To prove $v \in A^f$ is the same as to prove the inclusion $A \subset \mathrm{Val}_f(v)$. So, we get $\nu \in \mathrm{Val}_f(v), \forall \nu \in A$. We have

$$\mathrm{Val}_f(v) = (\neg s_{1*}\mathrm{Val}_f(u) \vee s_{2*}\mathrm{Val}_f(u)) \wedge$$
$$(s_{1*}\mathrm{Val}_f(u) \vee \neg s_{2*}\mathrm{Val}_f(u)).$$

Check that ν is included in the first parenthesis. Suppose this is not true.

Then $\nu \in s_{1*}\mathrm{Val}_f(u), \nu s_1 \in \mathrm{Val}_f(u)$. The equality $\nu s_1 = \nu s_2$ and the inclusions $\nu s_2 \in \mathrm{Val}_f(u), \nu \in s_{2*}\mathrm{Val}_f(u)$ lead to a contradiction. Hence, ν is included in the first parenthesis, and, analogously, in the second one. Therefore, $A \subset \mathrm{Val}_f(v)$, $v \in A^f$. Thus, $s_{1*}(u)$ and $s_{2*}(u)$ coincide modulo the filter A^f. This leads to $\mu_X s_{1*}(u) = \mu_X s_{2*}(u), \overline{s_{1*}}\mu_Y(u) = \overline{s_{2*}}\mu_Y(u), \overline{s_{1*}} = \overline{s_{2*}}$. This completes the proof of the proposition.

Let us discuss the converse statement. Let $\overline{s_{1*}} = \overline{s_{2*}}$. Then $\overline{s_{1*}}\mu_Y(u) = \overline{s_{2*}}\mu_Y(u), \forall u \in \mathrm{Hal}_{\Phi\Theta}(Y)$. Therefore, $\mu_X s_{1*}(u) = \mu_X s_{2*}(u)$. Hence, $s_{1*}(u)$ and $s_{2*}(u)$ are equivalent modulo A^f.

Consider the same $v = (\neg s_{1*}(u) \vee s_{2*}(u)) \wedge (s_{1*}(u) \vee \neg s_{2*}(u)) \in A^f, A \subset \mathrm{Val}_f(v)$. As before, $\nu \in A$ implies $\nu \in \mathrm{Val}_f(v)$ for every u.

Now we are interested in the interpretation of the last inclusion. Let $\nu \in \neg s_{1*}\mathrm{Val}_f(u)$, $\nu s_1 \notin \mathrm{Val}_f(u)$. Using the second parenthesis in the record of v, we come to $\nu s_2 \notin \mathrm{Val}_f(u)$.

If $\nu s_1 \in \mathrm{Val}_f(u)$, then the inclusion $\nu s_2 \in \mathrm{Val}_f(u)$ follows from the first parenthesis. Does this lead to $\nu s_1 = \nu s_2$? The problem arising here is that if

$$\exists \nu \in A,\ \nu s_1 \neq \nu s_2,\ \exists u \in \mathrm{Hal}_{\Phi\Theta}(Y),$$

with the properties $\nu s_1 \in \mathrm{Val}_f(u), \nu s_2 \notin \mathrm{Val}_f(u)$. In other words, if there exists u which separates two different points νs_1 and νs_2? It is not true in general. However, in the case when in the corresponding Zarisski topology every point is f-closed this property holds. In this case we have the statement, opposite to Proposition 7. Recall also that the opposite statement is true in the equational geometry.

The objects of the category $\mathbb{C}_{\Phi\Theta}(f)$ are of the form $\mathrm{Hal}_{\Phi\Theta}(X)/T$, where T is f-closed Boolean filter in $\mathrm{Hal}_{\Phi\Theta}(X)$. The morphisms are homomorphisms of such

algebras, induced by homomorphisms $s\colon W(X) \to W(Y)$. As before, they are of the type $\overline{s}_*$, i.e., not arbitrary homomorphism of the given Boolean algebras. The transition

$$(X, A) \to \mathrm{Hal}_{\Phi\Theta}(X)/A^f$$

determines a contravariant functor

$$K_{\Phi\Theta}(f) \to \mathbb{C}_{\Phi\Theta}(f)$$

for exact $K_{\Phi\Theta}(f)$, which follows from the proposition 6. This functor is not a duality in general. Consider now a special case when it is a duality.

Let Θ be a variety of algebras and let G be an algebra in Θ. Consider the variety $\Theta(G)$ of algebras in Θ with the constants from G. For example, if Θ is the variety of commutative and associative rings with the unit and P is a field, then $\Theta(P) = Var - P$ is the variety of algebras over P.

In the general case we consider the G-algebra G in $\Theta(G)$ and the equations in it. Here, the free algebra is of the type $W(X) = G * W_0(X)$, where $W_0(X)$ is free in Θ, $*$ is the free product in Θ.

Let $\mu : W(X) \to G$ be a point, $X = \{x_1, \dots, x_n\}$, and $\mu(x_1) = a_1, \dots, \mu(x_n) = a_n$. The constants $a_1, \dots, a_n$ are considered as elements in $W(X)$. Thus, we can speak of a formula u of the type

$$(x_1 \equiv a_1) \wedge \cdots \wedge (x_n = a_n).$$

In this case $\mathrm{Val}_f(u)$ consists only of the element μ, and the point μ is closed in the Zariski topology. Here we have the duality of categories.

Note also that the point μ is a simple variety, determined by elementary formula u. Let us make some remarks, related to this observation.

Let u be a formula of the type X and v be an elementary formula of the type $Y \supset X$, equivalent to v. Let us associate algebraic varieties $\mathrm{Val}_f(u)$ and $\mathrm{Val}_f(v)$. Proceed from the identical inclusion $s^0\colon W(X) \to W(Y)$ and $v = s^0_* u$. We have:

$$\mathrm{Val}_f(v) = \mathrm{Val}_f(s^0_* u) = s^0_* \mathrm{Val}_f(u).$$

Here $\mathrm{Val}_f(v)$ is a cylinder in $\mathrm{Hom}(W(Y), G)$, over the initial variety $\mathrm{Val}_f(u)$. This new cylinder is determined by the elementary formula.

Consider further one special characteristic of the objects in the category $\mathbb{C}_{\Phi\Theta}(X)$. These are Boolean algebras of the form $\mathrm{Hal}_{\Phi\Theta}(X)/A^f$.

We have a canonical homomorphism

$$\mathrm{Val}_f^X\colon \mathrm{Hal}_{\Phi\Theta}(X) \to \mathrm{Bool}(W(X), G)$$

for every $X \in \Gamma^0$. We have also an isomorphism of Boolean algebras.

$$\chi : \mathrm{Bool}(W(X), G) \to Fun(\mathrm{Hom}(W(X), G), Z).$$

Here $Z = \{0, 1\}$ is a two-element Boolean algebra, $Fun(\mathrm{Hom}(W(X), G), Z)$ is a Boolean algebra of binary functions, and $\chi(A)$ is the characteristic function of the set A in $\mathrm{Hom}(W(X), G)$.

Consider further the composition of homomorphisms

$$\sim = (\chi \mathrm{Val}_f^X)\colon \mathrm{Hal}_{\Phi\Theta}(X) \to Fun(\mathrm{Hom}(W(X), G), Z).$$

Here $\tilde{u}$ is a characteristic function of the algebraic set $\mathrm{Val}_f(u)$, corresponding to a formula $u \in \mathrm{Hal}_{\Phi\Theta}(X)$.

Take an algebraic set $A = T^f$ with $T = A^f$ for the given model (G, Φ, F), and consider a mapping

$$\psi_A\colon Fun(\mathrm{Hom}(W(X), G), Z) \to Fun(A, Z)$$

which associates a restriction of the function on A to every function from the left side. It is a homomorphism of Boolean algebras.

Let $\tilde{u}_A = \psi_A(\tilde{u})$. The transition $u \to \tilde{u}$ is a homomorphism of Boolean algebras. Call its image *an algebra of regular functions on the set A*.

Proposition 8. *The Boolean algebra $\mathrm{Hal}_{\Phi\Theta}(X)/T$ is isomorphic to the algebra of regular functions defined on the variety A.*

Proof. For every u and $\mu\colon W \to G$ we have $\tilde{u}(\mu) = 1$ if and only if $\mu \in \mathrm{Val}_f(u)$ or, the same, $u \in \mathrm{LogKer}(\mu)$. Now, $u \in T = A^f = \bigcap_{\mu \in A} \mathrm{LogKer}(\mu)$ if and only if $\tilde{\mu} = 1$ for every $\mu \in A$. This means that $u \in T$ if and only if $\tilde{u}_A$ is a unit in the considered algebra of regular functions on the set A. This implies the proposition.

Further we need the following definition:

Definition. *Two models* (G_1, Φ, f_1) *and* (G_2, Φ, f_2) *are geometrically equivalent if and only if for the arbitrary set of formulas* T *of some type* X *we have*

$$T^{f_1 f_1} = T^{f_2 f_2}.$$

Proposition 9. *If the models* (G_1, Φ, f_1) *and* (G_2, Φ, f_2) *are geometrically equivalent then the (weak) categories* $K_{\Phi\Theta}(f_1)$ *and* $K_{\Phi\Theta}(f_2)$ *are isomorphic.*

Proof. This proposition for the exact categories in the equational theory follows directly from the corresponding duality. In our case there is no duality and we need a proof.

Let the given models (G_1, Φ, f_1) and (G_2, Φ, f_2) be geometrically equivalent. Build an isomorphism

$$F\colon K_{\Phi\Theta}(f_1) \to K_{\Phi\Theta}(f_2).$$

Define $F(A) = A^{f_1 f_2} = B$ for every object A in the category $K_{\Phi\Theta}(f_1)$. If A is of the type X, then B is of the same type. Similarly, $B^{f_2 f_1}$ for B. Here, $A^{f_1 f_2 f_2 f_1} = A^{f_1 f_1 f_1 f_1} = A^{f_1 f_1} = A$. This means that F is a bijection on the objects. Let now $s\colon W(Y) \to W(X)$ be given. Check that this s is admissible for A_1 and A_2 in $K_{\Phi\Theta}(F_1)$ if and only if this s is admissible for the corresponding B_1 and B_2 in $K_{\Phi\Theta}(f_2)$. Let $A_1 \subset sA_2$ be given. Then $A_1^{f_1} \supset (sA_2)^{f_1}$ and $A_1^{f_1 f_2} \subset (sA_2)^{f_1 f_2}$. We have:

$$(sA_2)^{f_1} \supset s^* A_2^{f_1},$$
$$(sA_2)^{f_1 f_2} \subset (s^* A_2^{f_1})^{f_2} = sA_2^{f_1 f_2}.$$

Hence,

$$B_1 \subset (sA_2)^{f_1 f_2} \subset sB_2.$$

This means that s is admissible for B_1 and B_2.

The opposite direction can be checked similarly. The proposition is proved.

Let us make a remark on the notion of geometrical equivalence of models.

Proposition 10. *If two models are geometrically equivalent, then they are elementary equivalent.*

Proof. Let the models (G_1, Φ, f_1) and (G_2, Φ, f_2) be geometrically equivalent, and let the formula u of some type X belongs to the elementary theory of the first model. We have $u \in \mathrm{Hal}_{\Phi\Theta}(X)$ and $\mathrm{Val}_{f_1}(u) = \mathrm{Hom}(W(X), G_1)$.

Let $A = \mathrm{Hom}(W(X), G_1)$ and $T = A^{f_1}$. Obviously, $T = T(X)$ is the elementary theory of the type X of the first model and $u \in T$. The set T is f_1-closed. Moreover, it is the minimal one with this property in $\mathrm{Hal}_{\Phi\Theta}(X)$. By assumption, T is also f_2-closed and is the minimal one with this property. Hence, if $B = \mathrm{Hom}(W(X), G_2)$, then $B^{f_2} = T$. Therefore, T is an elementary theory of the type X of the second model.

Since $u \in T$ is the formula, it belongs to the elementary theory of the second model. Analogously, if the formula u belongs to the elementary theory of the second model, then u holds in the first model.

Problem 1. *Is the opposite true?, i.e., is it true that elementary equivalent models are geometrically equivalent?*

Note in the conclusion that the category $K_\Theta(G)$ in the equational theory is a full subcategory in $K_{\Phi\Theta}(f)$ for the given model (G, Φ, f). There is no such a relation for $C_\Theta(G)$ and $C_{\Phi\Theta}(f)$. The objects of the first one are algebras in Θ, while the objects of the second one are Boolean algebras.

5.5. The categories $K_{\Phi\Theta}$ and $C_{\Phi\Theta}$.

The objects of $K_{\Phi\Theta}$ are of the form

$$(X, A, (G, \Phi, f)).$$

Here, the model (G, Φ, f) is not fixed in the category, and $A = T^f$ for some T in $\mathrm{Hal}_{\Phi\Theta}(X)$.

The objects of $C_{\Phi\Theta}$ are of the form

$$(\mathrm{Hal}_{\Phi\Theta}(X)/T, (G, \Phi, f)),$$

where T is f-closed Boolean filter in $\mathrm{Hal}_{\Phi\Theta}(X)$.

Let us pass to morphisms. Given a homomorphism $\delta : G_1 \to G_2$ in Θ, we have $\tilde{\delta} : \mathrm{Hom}(W(X), G_1) \to \mathrm{Hom}(W(X), G_2)$ by the rule $\tilde{\delta}(\nu) = \delta\nu$. According to the commutative diagram

$$\begin{array}{ccc} W(Y) & \xrightarrow{s} & W(X) \\ {\scriptstyle \nu'}\downarrow & & \downarrow{\scriptstyle \nu} \\ G_2 & \xleftarrow{\delta} & G_1 \end{array}$$

we can write $\nu' = \delta\nu s = \tilde{\delta}(\nu s) = \tilde{\delta}\tilde{s}(\nu) = \tilde{s}\tilde{\delta}(\nu) = (s, \delta)(\nu)$.

Let now two objects $(X, A, (G_1, \Phi, f_1))$ and $(Y, B, (G_2, \Phi, f_2))$ be given. We say that the pair (s, δ) is admissible for A and B if $(s, \delta)(\nu) \in B$ for every $\nu \in A$. Such admissible (s, δ) determines morphisms in the weak category $K_{\Phi\Theta}$.

For every set B of the type Y let us consider a set $(s, \delta)B$ of the type X, determined by the rule:

$$\nu : W(X) \to G_1 \in (s, \delta)B \text{ iff } (s, \delta)(\nu) \in B.$$

We have: $\tilde{s}(\tilde{\delta}(\nu)) \in B$ and $\tilde{\delta}(\nu) \in sB$, $\nu \in \delta sB$. Here sB is of the type X. If C is a set of a type Y, then δC is of the same type, and $\nu \in \delta C$ if $\tilde{\delta}(\nu) = \delta\nu \in G$. Hence, $(s, \delta)B = \delta sB = s\delta B$. Besides, the pair (s, δ) is admissible for A and B if and only if $A \subset s\delta B$. We write also $\delta B = \delta_* B$ and consider δ^* determined by $\delta^* B = \{\delta\nu | \nu \in B\}$.

We can define also exact morphisms of the type $([s], \delta)$. Given $\delta : G_1 \to G_2$, we have a mapping $[s] : A \to B$ by the rule $[s](\nu) = \delta\nu s, \nu \in A$. As a morphism we take a pair $([s], \delta)$. Morphisms

$$(\overline{s}, \delta) : (\mathrm{Hal}_{\Phi\Theta}(Y)/T_2,\ (G_2, \Phi, f_2)) \to$$
$$(\mathrm{Hal}_{\Phi\Theta}(X)/T_1, (G_1, \Phi, f_1))$$

are determined by the suitable pairs (s, δ). Like $([s], \delta)$, the homomorphism $\overline{s}$ of Boolean algebras depends on $\delta : G_1 \to G_2$. We will return to it further.

Consider now some details.

Given $A \subset \mathrm{Hom}(W(X), G_1)$, take $(\delta_* A)^{f_2}$ in $\mathrm{Hal}_{\Phi\Theta}(X)$.

Proposition 11. *The pair (s, δ) is admissible for the given A and B if and only if $su \in (\delta^* A)^{f_2}$ for every $u \in B^{f_2}$.*

Proof. Note first that for (s, δ) we have a homomorphism $\overline{s} : \mathrm{Hal}_{\Phi\Theta}(Y)/B^{f_2} \to \mathrm{Hal}_{\Phi\Theta}(X)/(s^* A)^{f_2}$. It is a morphism in the category $C_{\Phi\Theta}(f_2)$. Let (s, δ) be admissible for A and B. We claim that if $u \in T_2 = B^{f_2}$ then $su \in (\delta^* A)^{f_2}$.

The inclusion $u \in T_2 = B^{f_2}$ means that $B \subset \mathrm{Val}_{f_2}(u)$. Take an arbitrary ν in A. Then $\nu' = \delta\nu s \in B$ and $\delta\nu s \in \mathrm{Val}_{f_2}(u)$, $\delta\nu \in \mathrm{Val}_{f_2}(su)$. So, $\delta^* A \subset \mathrm{Val}_{f_2}(su)$, $su \in (\delta^* A)^{f_2}$.

Let now $su \in (\delta^* A)^{f_2}$ for every $u \in T_2$. Then $\nu' = \delta\nu s \in B$ for every $\nu \in A$. This means also that $\delta\nu \in sB = sT_2^{f_2} = (s^* T_2)^{f_s} = \bigcap_{v \in s^* T_2} \mathrm{Val}_{f_2}(v)$. We should

check that $\delta\nu \in \mathrm{Val}_{f_2}(v)$ holds for every $v \in s^*T_2$. We have $v = su, u \in T_2$. Hence, $v = su \in (s^*A)^{f_2}$ and $\delta^*A \subset \mathrm{Val}_{f_2}(v)$. Thus, $\delta\nu \in \delta^*A, \delta\nu \in \mathrm{Val}_{f_2}(v)$. The proposition is proved.

Consider once more the pairs (s, δ), admissible for the fixed A and B. Fix also $\delta : G_1 \to G_2$ and vary the component s. For every such s we have $[s] : A \to B$ and $\overline{s} : \mathrm{Hal}_{\Phi\Theta}(Y)/B^{f_2} \to \mathrm{Hal}_{\Phi\Theta}(X)/(\delta^*A)^{f_2}$.

Proposition 12. *For every given δ we have $[s_1] = [s_2] \to \overline{s_1} = \overline{s_2}$.*

Proof. The proof of this proposition repeats one of the proposition 6 and we omit it.

Further we return to morphisms in the category $C_{\Phi\Theta}$. As we have seen, they have the form

$$(\overline{s}, \delta) : (\mathrm{Hal}_{\Phi\Theta}(Y)/T_2, (G_2, \Phi, f_2)) \to$$
$$\to (\mathrm{Hal}_{\Phi\Theta}(X)/T_1, (G_1, \Phi, f_1)).$$

The pairs s and δ are correlated in a special way. Here, $\delta : G_1 \to G_2$ is a homomorphism of algebras in Θ. Take

$$(\delta^* T_1^{f1})^{f_2} = T_1^{\delta}$$

and assume that $s : W(Y) \to W(X)$ induces a homomorphism of Boolean algebras

$$\overline{s} : \mathrm{Hal}_{\Phi\Theta}(Y)/T_2 \to \mathrm{Hal}_{\Phi\Theta}(X)/T_1^{\delta}$$

which is a morphism in the category $C_{\Phi\Theta} f_2$. These conditions mean correlation between s and δ. This, in turn, implies that the pair (s, δ) is admissible for $A = T_1^{f_1}$ and $B = T_2^{f_2}$.

Pay attention to the fact that the homomorphism $\overline{s}$, is not in general a homomorphism of the initial Boolean algebras.

Let us consider multiplication of morphisms in $K_{\Phi\Theta}$ and $C_{\Phi\Theta}$ in more detail using the arguments above. Let

$$([s], \delta) : (X, A; (G_1, \Phi, f_1)) \to (Y, B; (G_2, \Phi, f_2))$$

and

$$([s'], \delta') : (Y, B; (G_2, \Phi, f_2)) \to (Z, C; (G_3, \Phi, f_3))$$

be given in the exact category $K_{\Phi\Theta}$. Define the multiplication:

$$([s'],\delta')([s],\delta) = ([ss'],\delta'\delta) : (X,A;(G_1,\Phi,f_1)) \to (Z,C;(G_3,\Phi,f_3)).$$

One should check that the homomorphisms ss' and $\delta'\delta$ are correctly coordinated. This means that $\delta'\delta\nu ss' \in C$ holds for every $\nu \in A$. We have $\delta\nu s \in B$ and, further, $\delta'(\delta\nu s)s' \in C$. Hence, $ss'(u) \in T_1^{(\delta'\delta)}$ for every $u \in T_3$. This gives the homomorphism

$$\overline{ss'} : \mathrm{Hal}_{\Phi\Theta}(Z)/T_3 \to \mathrm{Hal}_{\Phi\Theta}(X)/T_1^{(\delta'\delta)}.$$

Define the multiplication of morphisms in $C_{\Phi\Theta}$ as

$$(\overline{s}',\delta')(\overline{s},\delta) = (\overline{ss'},\delta'\delta).$$

We see that the transition $(X,A;(G,\Phi,f)) \to (\mathrm{Hal}_{\Phi\Theta}(X)/A^f,(G,\Phi,f))$ determines a contravariant functor $K_{\Phi\Theta} \to C_{\Phi\Theta}$ for weak and exact category $K_{\Phi\Theta}$.

Prove that the homomorphism $\overline{ss'}$ may be represented as a product of two morphisms in the category $C_{\Phi\Theta}(f_3)$ related to s and s'.

We have a diagram for s':

$$\begin{tikzcd}
Hal_{\Phi\Theta}(Z) \arrow[r, "s'"] \arrow[d, "\mu_Z"'] & \mathrm{Hal}_{\Phi\Theta}(Y) \arrow[d, "\mu_Y"] \\
\mathrm{Hal}_{\Phi\Theta}(Z)/T_3 \arrow[r, "\overline{s'}"] & \mathrm{Hal}_{\Phi\Theta}(Y)/T_2^{\delta'}
\end{tikzcd}$$

and another one for s

$$\begin{tikzcd}
\mathrm{Hal}_{\Phi\Theta}(Y) \arrow[r, "s"] \arrow[d, "\mu_Y"'] & \mathrm{Hal}_{\Phi\Theta}(X) \arrow[d, "\mu_X"] \\
\mathrm{Hal}_{\Phi\Theta}(Y)/T_2 \arrow[r, "\overline{s}"] & \mathrm{Hal}_{\Phi\Theta}(X)/T_1^{\delta}
\end{tikzcd}$$

The third diagram is built on the base of the second one:

$$\begin{tikzcd}
\mathrm{Hal}_{\Phi\Theta}(Y) \arrow[r, "s"] \arrow[d, "\mu'_Y"'] & \mathrm{Hal}_{\Phi\Theta}(X) \arrow[d, "\mu'_X"'] \\
\mathrm{Hal}_{\Phi\Theta}(Y)/T_2^{\delta'} \arrow[r, "\overline{s}^{\delta'}"] & \mathrm{Hal}_{\Phi\Theta}(X)/T_1^{(\delta'\delta)}
\end{tikzcd}$$

with $\overline{ss'} = \overline{s}^{\delta'}\overline{s'}$. Here $\overline{s}^{\delta'}$ and $\overline{s'}$ are morphisms in $C_{\Phi\Theta}(f_3)$.

We need to show that $su \in T_1^{(\delta'\delta)}$ holds for every $u \in T_2^{\delta'}$. The filter $T_2^{\delta'}$ is constructed with the use of the commutative diagram

$$\begin{tikzcd} & W(Y) \arrow[dl, "\nu'"'] \arrow[dr, "\mu"] & \\ G_3 & & G_2 \arrow[ll, "\delta'"'] \end{tikzcd}$$

Here $T_2^{\delta'} = (\delta'^*B)^{f_3} \subset \mathrm{Hal}_{\Phi\Theta}(Y)$.

Let us pass to $(T_2^{\delta'})^{f_3} = (\delta'^*B)^{f_3f_3} = C_1$.

Consider the diagram

$$\begin{tikzcd} & & W(X) \arrow[dll, "\nu''"'] \arrow[dl, "\nu'"] \arrow[d, "\nu"] \\ G_3 & G_2 \arrow[l, "\delta'"'] & G_1 \arrow[l, "\delta"'] \end{tikzcd}$$

It determines $T_1^{(\delta'\delta)} \subset \mathrm{Hal}_{\Phi\Theta}(X)$. The inclusion $su \in T_1^{(\delta'\delta)}$ for every $u \in T_2^{\delta'}$ means that the pair $(s, \delta'\delta)$ is admissible for A and C_1. Check the last statement. Let $\nu \in A$. Then $\delta\nu s \in B$ and, hence, $\delta'\delta\nu s \in \delta'^*B \subset (\delta'^*B)^{f_3f_3} = C_1$.

It is left to check the equality $\overline{ss'} = \overline{s}^{\delta'}\overline{s'}$. Using commutative diagrams

$$\begin{tikzcd} \mathrm{Hal}_{\Phi\Theta}(Z) \arrow[r, "s'"] \arrow[d, "\mu_Z"'] & \mathrm{Hal}_{\Phi\Theta}(Y) \arrow[r, "s"] \arrow[d, "\mu'_Y"'] & \mathrm{Hal}_{\Phi\Theta}(X) \arrow[d, "\mu'_X"] \\ \mathrm{Hal}_{\Phi\Theta}(Z)/T_3 \arrow[r, "\overline{s'}"] & \mathrm{Hal}_{\Phi\Theta}(Y)/T_2^{\delta'} \arrow[r, "\overline{s}^{\delta'}"] & \mathrm{Hal}_{\Phi\Theta}(X)/T_1^{(\delta'\delta)} \end{tikzcd}$$

and

$$\begin{tikzcd} \mathrm{Hal}_{\Phi\Theta}(Z) \arrow[r, "ss'"] \arrow[d, "\mu_Z"'] & \mathrm{Hal}_{\Phi\Theta}(X) \arrow[d, "\mu'_X"'] \\ \mathrm{Hal}_{\Phi\Theta}(Z)/T_3 \arrow[r, "\overline{ss'}"] & \mathrm{Hal}_{\Phi\Theta}(X)/T_1^{\delta'\delta} \end{tikzcd}$$

we can rewrite the product of morphisms as

$$(\overline{s'}, \delta')(\overline{s}, \delta) = (\overline{s}^{\delta'}\overline{s'}, \delta'\delta).$$

Let us return to the definition of the morphisms in $K_{\Phi\Theta}$. Given the morphism

$$(s, \delta) : (X, A; (G_1, \Phi, f_2)) \to (Y, B; (G_2, \Phi, f_2)),$$

we have $\delta\nu s \in B$ for every $\nu \in A$. Pass to the set δ^*A, i.e., all $\delta\nu, \nu \in A$. Here, $\delta^*A \subset sB$. The set sB is f_2-closed. Hence, if $A_1 = (\delta^*A)^{f_2f_2}$, then $A_1 \subset sB$ and the mapping $s : W(Y) \to W(X)$ is admissible for f_2-closed sets A_1 and B. We have here a morphism

$$[s] : (X, A_1) \to (Y, B)$$

in the category $K_{\Phi\Theta}(f_2)$. This morphism is coordinated with the morphism

$$\overline{s} : \mathrm{Hal}_{\Phi\Theta}(Y)/T_2 \to \mathrm{Hal}_{\Phi\Theta}(X)/T_1^{\delta}$$

in the category $C_{\Phi\Theta}(f_2)$. The definitions of morphisms in the categories $C_{\Phi\Theta}$ and $K_{\Phi\Theta}$ are now coordinated.

Note further that every category $K_{\Phi\Theta}(f)$ is a subcategory in $K_{\Phi\Theta}$ and every $C_{\Phi\Theta}(f)$ is also a subcategory in $C_{\Phi\Theta}$. Besides, we have subcategories $K_{\Phi\Theta}(G)$ in the category $K_{\Phi\Theta}$ for every given algebra $G \in \Theta$ and with arbitrary interpretations f of the set Φ in G. There are subcategories $C_{\Phi\Theta}(G)$ in the category $C_{\Phi\Theta}$.

Some more details on these categories. Objects of the category $K_{\Phi\Theta}(G)$ can be represented in the form

$$(X, A, f),$$

since G and Θ are fixed for this category. Homomorphisms $\delta : G \to G$ are identical homomorphisms.

Morphisms in $K_{\Phi\Theta}(G)$ are

$$[s] : (X, A, f_1) \to (Y, B, f_2),$$

where s and the unit are coordinated in accordance with the general definition of the category $K_{\Phi\Theta}$. Here, $A = A_1^{f_1}, B = T_2^{f_2}$ for some T_1 in $\mathrm{Hal}_{\Phi\Theta}(X)$ and $T_2 = \mathrm{Hal}_{\Phi\Theta}(Y)$. We may also assume that $T_1 = A^{f_1}$ and $T_2 = B^{f_2}$. The corresponding set A_1 is now $A^{f_2 f_2}$ and T_1^{δ} is $T_1^{f_1 f_2} = A^{f_2}$.

Homomorphism $s : W(Y) \to W(X)$ induces a homomorphism of Boolean algebras

$$\overline{s} : \mathrm{Hal}_{\Phi\Theta}(Y)/B^{f_2} \to \mathrm{Hal}_{\Phi\Theta}(X)/A^{f_2}.$$

Every such s determines a mapping $[s] : A \to B$ and, simultaneously, a mapping $A_1 \to B$.

As it was pointed out, the subcategory R_f of simple varieties (i.e., varieties, determined by one-element sets T) is selected in the category $K_{\Phi\Theta}(f)$ for a given model (G, Φ, f). This R_f is also a Halmos algebra (subalgebra in $\mathrm{Hal}_{\Theta}(G)$) coinciding with the image of the homomorphism

$$\mathrm{Val}_f : \mathrm{Hal}_{\Theta}(\Phi) \to \mathrm{Hal}_{\Theta}(G).$$

We can take the homomorphism

$$\mathrm{Val}_f : \mathrm{Hal}_{\Theta}(\Phi) \to R_f$$

and consider inclusions on the level of objects:

$$\mathrm{Val}_f : \mathrm{Hal}_\Theta(\Phi) \to K_{\Phi\Theta}(f).$$

Every such inclusion determines transitions from the sets of formulas T to the corresponding objects of the category $K_{\Phi\Theta}(f)$.

Consider a new category with the objects $\mathrm{Val}_f : \mathrm{Hal}_{\Phi\Theta}(f) \to K_{\Phi\Theta}(f)$ and commutative diagrams

$$\begin{tikzcd}
\mathrm{Hal}_\Theta(\Phi) \arrow[r, "\mathrm{Val}_{f_1}"] \arrow[dr, "\mathrm{Val}_{f_2}"'] & K_{\Phi\Theta}(f_1) \arrow[d, "\gamma"] \\
 & K_{\Phi\Theta}(f_2)
\end{tikzcd}$$

as morphisms.

Here, γ is a functor of categories, and the commutativity of the diagram is understood on the level of objects.

Besides, f_1 is associated with the model (G_1, Φ, f_1), and f_2 with (G_2, Φ, f_2). Thus, also the algebras G vary.

The corresponding functor γ, subject to the condition of commutativity of the diagram, we call a *special functor*.

The functor γ is called *strong*, if γ induces the algebra homomorphism $R_{f_1} \to R_{f_2}$.

Every special γ is strong.

If (G_1, Φ, f_1) and (G_2, Φ, f_2) are geometrically equivalent, then the functor $\gamma = \gamma_{f_1}, f_2 = f$, defined above, is a special isomorphism $K_{\Phi\Theta}(f_1) \to K_{\Phi\Theta}(f_2)$.

Define now the semimorphisms in the considered category. They are of the form

$$\begin{tikzcd}
\mathrm{Hal}_{\Phi\Theta} \arrow[r, "\mathrm{Val}_{f_1}"] \arrow[d, "\sigma"'] & K_{\Phi\Theta}(f_1) \arrow[d, "\gamma"] \\
\mathrm{Hal}_{\Phi\Theta} \arrow[r, "\mathrm{Val}_{f_2}"] & K_{\Phi\Theta}(f_2)
\end{tikzcd}$$

where σ is an automorphism of the algebra $\mathrm{Hal}_{\Phi\Theta} = \mathrm{Hal}_\Theta(\Phi)$.

In this case γ induces some semihomomorphism of algebras $R_{f_1} \to R_{f_2}$. The main problem is to find conditions on the models which provide an isomorphism of the corresponding categories of algebraic sets. We have already seen that if the models are geometrically equivalent then this is the case. We may also claim that

in this situation the models are elementary equivalent and, in particular, if they are finite, then they are isomorphic.

More general information we get from the Galois theory, to which we pass right now.

6. The Galois-Krasner theory in algebraic logic and algebraic geometry

6.1. The group $Aut(\mathrm{Hal}_\Theta(G))$.

The theory presented in this and next sections goes back to the work of M. Krasner [Kr]. Later there were the papers [Be]; [Ba], [Pl1], and others. Here we follow the schemes from [Be1] and [Pl1]. Some new details are related to peculiarities of Halmos algebra $\mathrm{Hal}_\Theta(G)$.

Automorphism σ of the Halmos algebra $\mathrm{Hal}_\Theta(G)$ determines an automorphism σ_X of the quantorian X-algebra $\mathrm{Bool}(W(X), G)$ for every finite set $X \subset X^0$. All these automorphisms σ_X are naturally coordinated with the definition of $s \in S_\Theta$.

Show that some automorphism $\delta_* = \sigma \in Aut(\mathrm{Hal}_\Theta(G))$ corresponds to an automorphism $\delta \in AutG$.

Start from the construction used earlier. Let a homomorphism $\delta : G_1 \to G_2$ be given. Then

$$\tilde{\delta} : \mathrm{Hom}(W(X), G_1) \to \mathrm{Hom}(W(X), G_2).$$

If now $B \subset \mathrm{Hom}(W(X), G_2)$, then

$$A = \delta_* B = \tilde{\delta}^{-1}(B) \subset \mathrm{Hom}(W(X), G_1),$$

that is $\mu \in A = \delta_* B$ if and only if $\delta\mu \in B$. Thus, we have

$$\delta_* : \mathrm{Hal}_\Theta(G_2) \to \mathrm{Hal}_\Theta(G_1).$$

This δ_* is compatible with the Boolean structure and with the morphisms: $\delta_*(sB) = s\delta_*(B)$ for every $s : W(Y) \to W(X)$. Indeed, let ν lie in $\mathrm{Hom}(W(Y), G_1)$. Then $\nu \in \delta_*(sB)$ if and only if $\delta\nu \in sB$, $(\delta\nu)s \in B$, $(\delta\nu)s = \delta(\nu s) \in B$. Since the multiplication of morphisms is associative, this takes place if and only if $\nu s \in \delta_* B$ and $\nu \in s\delta_* B$. This gives also that if $s : W(Y) \to W(X)$ is admissible for A and B, then the same s is admissible for $\delta_*(A)$ and $\delta_*(B)$. Indeed, if $A \subset sB$, then

$\delta_*(A) \subset \delta_*(sB) = s\delta_*(B)$. But δ_* is not compatible with quantifiers in general. δ_* is an isomorphism of Halmos algebras if and only if $\delta : G_1 \to G_2$ is an isomorphism in Θ. In this case s is admissible for A and B if and only if this s is admissible for $\delta_*(A)$ and $\delta_*(B)$. Hence if $\delta \in Aut(G)$, then $\delta_* = \sigma \in Aut(\mathrm{Hal}_\Theta(G))$.

We have the representation of groups

$$Aut(G) \to Aut(\mathrm{Hal}_\Theta(G)).$$

The main result now is the following:

Theorem 7. *The representation $Aut(G) \to Aut(\mathrm{Hal}_\Theta(G))$ is an isomorphism of groups.*

Proof. We start the proof from the following statement. Let M be an arbitrary non-empty set, $\mathfrak{M} = \mathrm{Sub}M$ is the Boolean algebra of all subsets in M. Then every automorphism of the Boolean algebra $\mathfrak{M}$ determines a substitution on the set M, which induces this automorphism. See, for example, [Pl1], p. 338.

Here, if τ is a substitution on the set M, then the corresponding automorphism τ_* is defined by the rule

$$a \in \tau_* A = \tau A \text{ iff } \tau a \in A,\ A \subset M.$$

Fix further a finite set X and proceed from $M = M_X = \mathrm{Hom}(W(X), G)$. Every automorphism σ of the algebra $\mathrm{Hal}_\Theta(G)$ determines an automorphism σ_X of the Boolean algebra $\mathrm{Bool}(W(X), G)$, i.e., of the algebra $\mathfrak{M}_X$. This σ_X is given by a substitution τ_X on the set $M_X = \mathrm{Hom}(W(X), G)$ depending on σ and X.

Denote by Σ_X the group of all substitutions of the set M_X, by Σ_G the group of all substitutions of the set G and by $\tilde{\Sigma}_G = \Sigma_G^X$ its cartesian power. Consider an inclusion

$$\sim : \tilde{\Sigma}_G \to \Sigma_X.$$

This inclusion is defined as follows:

if $\zeta \in \tilde{\Sigma}_G$ then a substitution $\tilde{\zeta} \in \Sigma_X$ is defined by the rule: for each $\mu : W(X) \to G$, there correspond a homomorphism $\tilde{\zeta}(\mu) : W(X) \to G$ defined by: $\tilde{\zeta}(\mu)(X) = \zeta(X)(\mu(X))$ for every $x \in X$.

We would like to characterize the group $\tilde{\Sigma}_G$ as a subgroup in Σ_X.

Consider an auxiliary proposition.

Definition. *The substitution $\tau \in \Sigma_X$ is called correct if*

$$\mu(x) = \nu(x) \Leftrightarrow (\tau\mu)(x) = (\tau\nu)(x)$$

for every $\mu, \nu : W(X) \to G$ and every $x \in X$

Proposition 13. *A substitution τ is correct if and only if $\tau = \tilde{\zeta}$ for some $\zeta = \tilde{\Sigma}_G$.*

Proof. Let $\tau = \tilde{\zeta}$ for some $\zeta = \tilde{\Sigma}_G$. Check the correctness of τ. Take $x \in X$, $\mu, \nu : W(X) \to G$. Then $\tilde{\zeta}(\mu)(x) = \zeta(x)\mu(x); \tilde{\zeta}(\nu)(x) = \zeta(x)\nu(x)$. It is clear that the equality $\tilde{\zeta}(\mu)(x) = \tilde{\zeta}(\nu)(x)$ holds if and only if $\mu(x) = \nu(x)$.

Let not τ be an arbitrary correct substitution on the set $M_X = \mathrm{Hom}(W(X), G)$. Build $\zeta \in \tilde{\Sigma}_G$ with the condition $\tau = \tilde{\zeta}$.

Fix some $x \in X$ and define a substitution $\zeta(x)$ on the set G. Take an arbitrary $g \in G$, and $\mu : W(X) \to G$ with $\mu(x) = g$. Set:

$$\zeta(x)g = (\tau\mu)(x).$$

Since τ is correct, this definition does not depend on the choice of μ. If $\nu(x) = \mu(x)$, then $(\tau\nu)(x) = (\tau\mu)(x) = \zeta(x)(g)$. Check that the mapping $\zeta(x) : G \to G$ is a substitution.

Let us note first of all that if τ is a correct substitution, then the same is valid for τ^{-1}. Indeed, take $x \in X, \mu, \nu : W(X) \to G$. Denote $\mu_1 = \tau^{-1}\mu, \nu_1 = \tau^{-1}\nu, \mu = \tau\mu_1, \nu = \tau\nu_1$. We have: $(\tau\mu_1)(x) = (\tau\nu_1)(x) \Leftrightarrow (\mu_1(x) = \nu_1(x)$. This gives $\mu(x)\nu(x) \Leftrightarrow (\tau^{-1}\mu)(x) = (\tau^{-1}\nu)(x)$. The substitution τ^{-1} is correct.

Take now an arbitrary element $g_1 \in G$ and show that $\zeta(x)(g) = g_1$ for some $g \in G$. Let $\mu_1(x) = g_1, (\tau^{-1}\mu_1)(x) = g = \mu(x), \mu = \tau^{-1}\mu_1$. Then $(\tau\mu)(x) = \zeta(x)(g) = (\tau\tau^{-1}\mu_1)(x) = \mu_1(x) = g_1$.

It remains to verify that $\zeta(x)(g_1) = \zeta(x)(g_2)$ implies $g_1 = g_2$. Let $\mu_1(x) = g_1$, $\mu_2(x) = g_2$. We have $\zeta(x)(g_1) = (\tau\mu_1)(x) = \zeta(x)(g_2) = (\tau\mu_2)(x)$. The last gives $\mu_1(x) = \mu_2(x)$ and $g_1 = g_2$. We checked that $\zeta(x)$ is a substitution.

Take $\zeta \in \tilde{\Sigma}_G = \Sigma_G^X$. According to the rule: $\zeta(x)$ is determined by τ. Verify that $\tau = \tilde{\zeta}$. Take an arbitrary $\mu : W(X) \to G$ and check that $\tau\mu = \tilde{\zeta}\mu$. Take an arbitrary $x \in X$. Then $(\tau\mu)(x) = \zeta(x)\mu(x) = (\tilde{\zeta}\mu)(x)$. This finishes the proof of the proposition.

Proposition 14. *A substitution τ commutes with the quantifiers in* $\mathrm{Bool}(W(X),G)$ *if and only if τ is correct.*

Proof. We mean commutativity of the form

$$\tau\exists xA = \exists x\tau A$$

for $x \in X$ and $A \subset \mathrm{Hom}(W(X),G)$.

Let τ be correct, $\tau = \tilde{\zeta}, \zeta \in \tilde{\Sigma}_G$. Take $\mu \in \tau\exists xA = \tilde{\zeta}\exists xA$. We have $\tilde{\zeta}\mu \in \exists xA$. Choose $\nu \in A$ with $(\tilde{\zeta}\mu)(x') = \nu(x')$ for every $x' \in X, x' \neq x$. Let $\nu = \tilde{\zeta}\nu_1$. Since $\nu \in A$, then $\nu_1 \in \tilde{\zeta}A$. The equality $\tilde{\zeta}\mu(x') = \tilde{\zeta}\nu_1(x')$ gives $\mu(x') = \nu_1(x')$ for every $x' \neq x$, and thus $\mu \in \exists x\tilde{\zeta}A$.

Let now $\mu \in \exists x\tilde{\zeta}A$. Take ν_1 in $\tilde{\zeta}A$ with $\mu(x') = \nu_1(x')$ for $x' \neq x$. We have $(\tilde{\zeta}\mu)(x') = (\tilde{\zeta}\nu_1)(x')$. Here, $\nu = \tilde{\zeta}\nu_1 \in A$. Hence, $\tilde{\zeta}\mu \in \exists xA$, $\mu \in \tilde{\zeta}\exists xA$. We have checked commutativity for $\tau = \tilde{\zeta}$.

Let us check the opposite direction, i.e., the commutativity fulfills and we verify the correctness of the substitution τ.

Let μ and ν be two elements in $\mathrm{Hom}(W(X),G)$, and $\mu(x) = \nu(x)$ for some $x \in X$. We need to check that $(\tau\mu)(x) = (\tau\nu)(x)$ holds true. Denote by Y the set X without x (i.e., $Y = X \setminus x$). Denote the product of all $\exists y, y \in Y$ by $\exists(Y)$. Commutativity of τ with each $\exists y$ implies commutativity with $\exists(Y)$. Take, further, the set A consisting of one element ν, $A = \{\nu\}$. Then $\mu \in \exists(Y)\{\nu\}$. Now,

$$\tau\mu \in \tau^{-1}\exists(Y)\{\nu\} = \exists(Y)\tau^{-1}\{\nu\} = \exists(Y)\{\tau\nu\}.$$

Hence, $(\tau\mu)(x) = (\tau\nu)(x)$.

Here we used commutativity of τ^{-1} with quantifiers. Similarly we derive that $(\tau\mu)(x) = (\tau\nu)(x)$ implies $\mu(x) = \nu(x)$. Thus, τ is a correct substitution.

Hence, one can assume that every automorphism of a quantifier X-algebra $\mathrm{Bool}(W(X),G)$ is realized by a correct substitution of the set $\mathrm{Hom}(W(X),G)$, depending on the given automorphism.

Consider further coordination with the operations $s \in S_\Theta$.

Pick out a subgroup Σ_0 in $\tilde{\Sigma}_G = \Sigma_G^X$, consisting of constant ζ, i.e., $\zeta(x) = \delta \in \Sigma_G$ for all $x \in X$. A constant ζ corresponds to a $\delta \in \Sigma_G$. This determines the inclusion $\Sigma_G \to \tilde{\Sigma}_G = \Sigma_G^X$ whose image is Σ_0. Simultaneously we have an inclusion $\mathrm{Aut}(G) \to \Sigma_G^X$ for each $X \subset X^0$.

Denote the pointed inclusion for every X by $\tilde{}X$. If $\delta \in \mathrm{Aut}(G)$ or $\delta \in \Sigma_G$, then $\delta^{\sim X} \in \tilde{\Sigma}_G$. Further we vary the finite set $X \subset X^0$ and pass to the multi-sorted variant of sets and algebras. In particular, we consider multi-sorted set M with the components $M_X = \mathrm{Hom}(W(X), G)$.

The substitution τ of M is a function, determining a substitution τ^X of the set M_X for every X. Such τ is correct if all τ^X are correct, $\tau^X = \zeta^{\sim X}, \tau = \tilde{\zeta}$. Here ζ is also a function, determining an element ζ^X in Σ_G^X for every X. To every $\delta \in \Sigma_G$ corresponds $\delta^{\sim X}$ in Σ_G^X, which determines a substitution $\tilde{\delta}$ of the multi-sorted set M.

Consider homomorphisms $s\colon W(X) \to W(Y)$. An operation $s : \mathrm{Bool}(W(X), G) \to \mathrm{Bool}(W(Y), G)$ in the algebra $\mathrm{Hal}_\Theta(G)$ corresponds to a homomorphism s. The substitution τ of a multi-sorted set M commutes with all such s if for every A in $\mathrm{Hom}(W(X), G)$ holds $\tau^Y sA = s\tau^X A$.

Let $\tau = \tilde{\zeta}$ be an arbitrary substitution.

Proposition 15. *The substitution τ commutes with all $s \in S_\Theta$ if and only if $\tau = \tilde{\delta}$ where δ is an automorphism of the algebra G.*

Proof. We already know that every $\delta \in \mathrm{Aut}(G)$ acts in the algebra $\mathrm{Hal}_\Theta(G)$ as an automorphism. In particular, this means that the function $\tilde{\delta} = \tau$ for such δ has the needed properties.

Let now $\tau = \tilde{\zeta}$ commute with operations ot the type s. We want to check if $\tau = \tilde{\delta}$, where $\delta \in \mathrm{Aut}(G)$.

Show first that every ζ^X is a constant, and then notice that all ζ^X for different $X \subset X^0$ get the same value $\delta \in \Sigma_G$. We will do it simultaneously. Take $x \in X$ and $y \in Y$ and assume that $\zeta^X(x) \neq \zeta^Y(y)$. Select an element $a \in G$ with $\zeta^X(x)a \neq \zeta^Y(y)a$, and take $\mu : W(Y) \to G$ with $\mu(y) = a$. Let $s : W(X) \to W(Y)$ transfer x to y. Then

$$(\tau^Y \mu)(sx) = (\tau^Y \mu)(y) = \zeta^Y(y)\mu(y) =$$

$$= \zeta^Y(y)(a),$$

$$\zeta^X(x)(a) = \zeta^X(x)\mu(y) = \zeta^X(x)(\mu s)(x) =$$

$$= (\zeta^X(x)(\mu s))(x),$$

$$(\zeta^X(x)(\mu s))(x) = \tau^X(\mu s)(x).$$

Hence,

$$((\tau^Y\mu)s)(x) \neq \tau^X(\mu s)(x),$$

$$(\tau^Y\mu)(s) \neq \tau^X(\mu s).$$

We can rewrite it as

$$\tilde{s}(\tau^Y\mu) \neq \tau^X(\tilde{s}(\mu)).$$

Take now an arbitrary $A \subset \mathrm{Hom}(W(X), G)$, and let $\mu \in s\tau^X A$. Thus, $\mu s = \tilde{s}(\mu) \in \tau^X A$, $\tau^X(\mu s) \in A$, $\tau^X(\tilde{s}(\mu)) \in A$.

On the other hand, $\mu \in \tau^Y(sA)$ means that $(\tau^Y)(\mu) \in sA$, $\tilde{s}(\tau^Y\mu) \in A$. Since $\tau^X(\tilde{s}(\mu)) \neq \tilde{s}(\tau^Y\mu)$, it is clear that for some A there is no commutativity. It suffices to take one-element set A. Thus, if τ is not constant, then there is no commutativity. Hence, τ is constant and $\tau = \tilde{\delta}, \delta \in \Sigma_G$.

Prove further that the substitution δ is an automorphism of the algebra G if the commutativity takes place.

Let $\omega \in \Omega$ be an n-ary operation and $a_1, \dots, a_n$ elements in G. Check that

$$\delta(a_1 \dots a_n\omega) = (\delta a_1) \dots (\delta a_n)\omega.$$

Take $X = \{x, x_1, \dots, x_n\}$, $W = W(X)$ and $\mu : W \to G$ with the condition $\mu(x_1) = a_1, \dots, \mu(x_n) = a_n$. Take also $s : W \to W$ with $sx = x_1 \dots x_n\omega$. The commutativity with s means for $\tau = \tilde{\delta}$ that

$$\tau(\mu s) = \tau(\mu) \cdot s.$$

For the given x we have

$$\tau(\mu s)(x) = \tilde{\delta}(\mu s)(x) = \delta(\mu s)(x) =$$

$$= \delta(\mu s(x)) = \delta(\mu(x_1 \dots x_n\omega)) = \delta(a_1 \dots a_n\omega),$$

$$(\tau(\mu) \cdot s)(x) = \tau(\mu)(sx) = \tau(\mu)(x_1 \dots x_n)\omega =$$

$$(\tau\mu(x_1)) \dots (\tau\mu(x_n))\omega = \delta(a_1) \dots \delta(a_n)\omega.$$

This finishes the proof of Proposition 14. Return to the proof of Theorem 7.

Let σ be an automorphism of the algebra $\mathrm{Hal}_\Theta(G)$. This σ induces an automorphism of every Boolean algebra $\mathrm{Bool}(W(X), G)$. Thus, σ is determined by a substitution τ of the multi-sorted set M with the components $M(X) = \mathrm{Hom}(W(X), G)$. The commutativity with quantifiers means that $\tau = \tilde{\zeta}$, while commutativity with operations of $s \in S_\Theta$ type leads to $\tau = \tilde{\delta}$ where δ is an automorphism of the algebra G. It is easy to see that $\delta_* = \sigma$ for such δ. Theorem 7 is proved.

6.2 The main results.

Let a model (G, Φ, f) be given. A group of automorphisms $\mathrm{Aut}(f)$ of this model is a subgroup in a group $\mathrm{Aut}(G)$. As we have seen, $\mathrm{Aut}(G)$ acts in each set $\mathrm{Hom}(W(X), G)$. A group $\mathrm{Aut}(f)$ also acts there.

Theorem 8. *Every f-algebraic set in $\mathrm{Hom}(W(X), G)$ is invariant under the action of the group $\mathrm{Aut}(f)$.*

Proof. Check first that for every formula $u \in \mathrm{Hal}_{\Phi\Theta}(X)$, every point $\mu : W \to G$ and every automorphism $\delta \in \mathrm{Aut}(f)$ we have

$$\mu \in \mathrm{Val}_f(u) \Leftrightarrow \delta\mu \in \mathrm{Val}_f(u).$$

Let us call this property of the formula u the correctness property. We want to show that every u is correct.

Although this is trivial, we give the precise proof. All atomic formulas are correct, and it is easy to see that if u and v are correct, then $u \vee v$, $u \wedge v$, $\neg u$ are correct as well. Let u be correct. Take the formula $\exists xu$ for some $x \in X$. Let $\mu \in \mathrm{Val}_f(\exists xu) = \exists x\mathrm{Val}_f(u)$. We can take a point $\nu : W \to G$ such that $\nu \in \mathrm{Val}_f(u), \mu(y) = \nu(y)$ for every $y \in X, y \neq x$. Take now $\delta \in \mathrm{Aut}(f)$. We have: $\sigma\mu(y) = \delta\nu(y), \delta\nu \in \mathrm{Val}_f(u)$. So, $\delta\mu \in \exists x\mathrm{Val}_f(u) = \mathrm{Val}_f(\exists xu)$.

Let, conversely, $\delta\mu \in \mathrm{Val}_f(\exists xu)$. Take some $\nu = \delta\nu_1 \in \mathrm{Val}_f(u)$ with $\nu(y) = \delta\nu_1(y) = \delta\mu(y), y \neq x$. Then $\mu(y) = \nu_1(y), \nu_1 \in \mathrm{Val}_f(u)$ and $\mu \in \exists x\mathrm{Val}_f(u) = \mathrm{Val}_f(\exists xu)$.

Finally, let $v = su$, where u of some type Y is correct. Given $s : W(Y) \to W(X)$, we get $\mathrm{Val}_f(v) = s\mathrm{Val}(u)$. Take $\delta \in Aut(f)$ and let $\mu \in \mathrm{Val}_f(v), \mu \in s\mathrm{Val}_f(u), \mu s \in \mathrm{Val}_f(u)$. By the condition on u we have $\delta\mu s \in \mathrm{Val}_f(u), \delta\mu \in s\mathrm{Val}_f(u) = \mathrm{Val}_f(su) = \mathrm{Val}_f(v)$.

Let now $\delta\mu \in \mathrm{Val}_f(v) = s\mathrm{Val}_f(u)$ be given. Then $\delta\mu s \in \mathrm{Val}_f(u)$ and $\mu s \in \mathrm{Val}_f(u), \mu \in \mathrm{Val}_f(v)$.

So, all correct formulas form a subalgebra in $\mathrm{Hal}_\Theta(\Phi)$. This subalgebra contains the set of generators M, and thus, all formulas in $\mathrm{Hal}_\Theta(\Phi)$ are correct.

Let now $A = T^f = \cap_{u \in T} \mathrm{Val}_f(u)$ for some T of the type X, $\mu \in A, \delta \in \mathrm{Aut}(f)$. Then for every $u \in T$ we have $\mu \in \mathrm{Val}_f(u)$, $\delta\mu \in \mathrm{Val}_f(u), \delta\mu \in A$.

The theorem is proved.

Note in addition that if the set A is f-closed and $\delta \in \mathrm{Aut}(f)$, then $\delta_* A = \delta A = A$. Indeed, we may claim that if $\delta_* A = A$ for some $\delta \in \mathrm{Aut}(G)$ and $A \subset \mathrm{Hom}(W(X), G)$, then there is a model (G, Φ, f) with some Φ and f, such that $\delta \in \mathrm{Aut}(f)$ and the set A is f-closed.

Let us prove this claim.

Take $X = \{x_1, \ldots, x_n\}$. Let the set Φ consist of one n-ary relation φ. Consider a formula $\varphi(x_1, \ldots, x_n)$, and realize φ in G. We proceed from the standard bijection $\pi : \mathrm{Hom}(W(X), G) \to G^{(n)}$, setting $f(\varphi) = A^\pi \subset G^{(n)}$. We have $\mathrm{Val}_f(\varphi(x_1, \ldots, x_n)) = A$ for such f. Thus, A is an f-algebraic set. The condition $\delta A = A$ means that the automorphism δ is coordinated with the interpretation f of the given set Φ, i.e., $\delta \in \mathrm{Aut}(f)$.

Let us make some remarks on the constructions above. Given a model (G_1, Φ, f) and an algebra isomorphism $\delta : G_1 \to G_2$, consider another model (G_2, Φ, f^δ) with the commutative diagram

$$\begin{tikzcd} & \mathrm{Hal}_\Theta(\Phi) \arrow[dl, "\mathrm{Val}_{f^\delta}"'] \arrow[dr, "\mathrm{Val}_f"] & \\ \mathrm{Hal}_\Theta(G_2) \arrow[rr, "\delta_*"] & & \mathrm{Hal}_\Theta(G_1) \end{tikzcd}$$

We determine f^δ such that the models are isomorphic and the diagram is indeed commutative.

Take a formula u of the type X in $\mathrm{Hal}_\Theta(\Phi)$. It should be:

$$\mathrm{Val}_f(u) = \delta_* \mathrm{Val}_{f^\delta}(u).$$

Let φ be an n-ary relation in Φ. Take $X = \{x_1, \ldots, x_n\}$ and $u = u(x_1, \ldots, x_n)$. Pass to $\delta^{(n)} : G_1^{(n)} \to G_2^{(n)}$ and to bijections $\pi_1 : \mathrm{Hom}(W(X), G_1) \to G_1^{(n)}$ and $\pi_2 : \mathrm{Hom}(W(X), G_2) \to G_2^{(n)}$. The diagram

$$\begin{tikzcd} \mathrm{Hom}(W(X), G_1) \arrow[r, "\tilde{\delta}"] \arrow[d, "\pi_1"'] & \mathrm{Hom}(W(X), G_2) \arrow[d, "\pi_2"] \\ G_1^{(n)} \arrow[r, "\delta^{(n)}"] & G_2^{(n)} \end{tikzcd}$$

is commutative.

We should determine $f^\delta(\varphi)$, solving the equation $\mathrm{Val}_f(u) = \delta_* \mathrm{Val}_{f^\delta}(u)$ for $u = \varphi(x_1, \ldots, x_n)$. Denote $A = \mathrm{Val}_f(u)$ and $B = \mathrm{Val}_{f^\alpha}(u)$. Then $A = \delta_* B$. Here, $f(\varphi) = \pi_1^* A$ and $f^\delta(\varphi) = \pi_2^* B$. We have: $\nu \in A$ if and only if $\delta\nu \in B$, which gives

$(a_1, \ldots, a_n) \in f(\varphi)$ if and only if $(\delta a_1, \ldots, \delta a_n) \in f^\delta(\varphi)$. The last determines the model (G_2, Φ, f^δ), isomorphic to the initial model (G_1, Φ, f) by the isomorphism δ.

It can be checked that the second model realizes the commutative diagram.

Let us relate the algebraic sets for the models f and f^δ. Let $A = T^f, B = T^{f^\delta}$ for some $T \subset \mathrm{Hal}_{\Phi\Theta}(X)$. Then $A = \delta_* B$. Indeed,

$$A = T^f = \bigcap_{u \in T} \mathrm{Val}_f(u) = \cap_{u \in T} \delta_* \mathrm{Val}_{f^\delta}(u) =$$
$$= \delta_*(\bigcap_{u \in T} \mathrm{Val}_{f^\delta}(u)) = \delta_* T^{f^\delta} = \delta_* B.$$

Note that the isomorphic models are geometrically equivalent. This follows, for example, from the proposition 3. Hence, the models (G_1, Φ, f) and (G_2, Φ, f^δ) are geometrically equivalent. If $T = A^f$ then the filter T is simultaneously f and f^δ-closed. For such T we have

$$T = A^f = (\delta_* B)^f = B^{f^\alpha} = T, (\delta_* B)^f = B^{f^\delta}.$$

The theory we consider here is quite natural to call Galois-Krasner theory.

The algebra $\mathrm{Hal}_\Theta(G)$ is considered together with its group of automorphisms, represented as $\mathrm{Aut}G$. For every $A \in \mathrm{Hal}_\Theta(G)$ and $\sigma \in \mathrm{Aut}G$, we write σA instead of $\sigma_* A$. We give further standard definitions.

Let H be a subset in $\mathrm{Aut}G$. Then $R = H'$ is a subalgebra in $\mathrm{Hal}_\Theta(G)$, consisting of all elements A, for which $\sigma A = A$ for every $\sigma \in H$. Here $R = H'$ is actually a subalgebra of a multi-sorted algebra $\mathrm{Hal}_\Theta(G)$ and for every $X \in \Gamma^0$ the corresponding component $R(X)$ is a subalgebra of the quantifier X-algebra $\mathrm{Bool}(W(X), G)$. The algebra $R(X)$ contains all equalities of the type X.

Let now R be a subset (multi-sorted) of $\mathrm{Hal}_\Theta(G)$. Then $R' = A$ is a subgroup in $\mathrm{Aut}(G)$, consisting of all $\sigma \in \mathrm{Aut}(G)$ for which $\sigma A = A$ for every $A \in R$. It is actually a subgroup in $\mathrm{Aut}G$. We have Galois correspondence and Galois closure $H'' = (H')'$ and $R'' = (R')'$. A subgroup H is closed if $H'' = H$. Every closed subalgebra R contains all equalities in $\mathrm{Hal}_\Theta(G)$. Recall that the equality in $\mathrm{Hal}_\Theta(G)$ is an algebraic set determined by an equation of the $w \equiv w'$ type.

Theorem 9. *For every model (G, Φ, f) we have*

$$R'_f = \mathrm{Aut}(f).$$

Proof. It follows from Theorem 8 that the inclusion $\mathrm{Aut}(f) \subset R'_f$ takes place.

Let now $\sigma \in R'_f$. It means that $\sigma A = A$ for every $A \in R'_f$. We need to show that the automorphism σ of the algebra G is compatible with the interpretation $f(\varphi)$ of the arbitrary $\varphi \in \Phi$.

Let the relation φ be n-ary. Take $X = \{x_1, \ldots, x_n\}$ and consider the formula $\varphi(x_1, \ldots, x_n)$. Take $A = \mathrm{Val}_f(\varphi(x_1, \ldots, x_n))$. For the bijection $\pi : \mathrm{Hom}(W(X), G) \to G^{(n)}$ we have $A = \pi^{-1}(f(\varphi))$. Coordination of σ with $f(\varphi)$ is equal to the condition $\sigma A = A$. This condition holds true, since $A \in R_f$. Thus, $\sigma \in \mathrm{Aut}(f)$ and $R'_f = \mathrm{Aut}(f)$.

We cannot claim that $(\mathrm{Aut}(f))' = R_f$ holds. Indeed, in the general case we can consider an algebraic set A which is not simple, is not determined by a single formula u and, hence, does not belong to the algebra R_f. However, $\sigma A = A$ for every $\sigma \in \mathrm{Aut}(f)$ by Theorem 8. Thus, $R_f \subset \mathrm{Aut}(f)'$ and the inclusion may be proper.

Let us formulate the first main theorem.

Theorem 10. *If the algebra G is finite, then every subgroup in $\mathrm{Aut}(G)$ is closed and every subalgebra R in $\mathrm{Hal}_\Theta(G)$, containing all equalities, is also closed.*

We have here a bijection between such R and subgroups in $\mathrm{Aut}(G)$.

We give the proof of this theorem in the next subsection.

Theorem 11. *If the algebra G is finite, then for every model (G, Φ, f) we have*

$$\mathrm{Aut}(f)' = R_f.$$

Proof. From Theorems 9 and 10 follows that

$$R''_f = R_f = (R'_f)' = \mathrm{Aut}(f)'.$$

It follows from Theorem 11 that if an algebra G is finite, then every f-algebraic set A for the model (G, Φ, f) is a simple algebraic set. Now we can state that algebraic varieties possesses the property to be noetherian . This can also be proved directly if the algebra G is finite.

Consider the second main Galois theorem.

Take an isomorphism $\delta : G_2 \to G_1$ of algebras in Θ. An isomorphism of Halmos algebras

$$\delta_* : \mathrm{Hal}_\Theta(G_1) \to \mathrm{Hal}_\Theta(G_2)$$

corresponds to δ.

The groups $\mathrm{Aut}(G_1)$ and $\mathrm{Aut}(G_2)$ are isomorphic. Their isomorphism is determined by $\sigma \to \delta\sigma\delta^{-1}$, $\sigma \in \mathrm{Aut}(G_2)$. This gives $\mathrm{Aut}(G_2) \to \mathrm{Aut}(G_1)$.

Let now R_1 be a subalgebra in $\mathrm{Hal}_\Theta(G)$. We have $\delta_*(R_1) = R_2$ in $\mathrm{Hal}_\Theta(G)$. Hence, R_1 and R_2 are isomorphic. We check directly

$$R_1' = \delta R_2' \delta^{-1} (\delta_*(R_1))' = \delta^{-1} R_1' \delta.$$

In this sense, the subgroups R_1' in $\mathrm{Aut}(G_1)$ and R_2' in $\mathrm{Aut}(G_2)$ are conjugated by the isomorphism $\delta : G_2 \to G_1$.

Theorem 12. *If algebras G_1 and G_2 are finite, R_1 is a subalgebra in* $\mathrm{Hal}_\Theta(G_1)$ *and R_2 is a subalgebra in* $\mathrm{Hal}_\Theta(G_2)$, *then every isomorphism $\gamma : R_1 \to R_2$ is induced by an isomorphism $\delta : G_2 \to G_1$. We have $\gamma = \delta_* : R_1 \to R_2$.*

6.3. Proofs.

We assume that a finite algebra G is considered also as a subset in X^0. A free in Θ algebra $W(G)$ is an object of the category Θ^0. Consider an affine space $\mathrm{Hom}(W(G), G)$ and a quantifier G-algebra $\mathrm{Bool}(W(G), G)$ as an object of the Halmos category $\mathrm{Hal}_\Theta(G)$.

Let us make remarks on the set $\mathrm{Hom}(W(G), G)$ and the algebra $\mathrm{Bool}(W(G), G)$.

We have a standard bijection

$$\pi : \mathrm{Hom}(W(G), G) \to G^G,$$

where $\pi(\nu) : G \to G$ is a restriction of the homomorphism ν on the set G.

The right part, G^G, is a semigroup of transformations of the set G. Let us extend the multiplication in this semigroup to the left part. If ν_1, ν_2 are two elements in $\mathrm{Hom}(W(G), G)$, then $\nu_1\nu_2 : W(G) \to G$ is a homomorphism, determined by the condition $\pi(\nu_1\nu_2) = \pi(\nu_1) \cdot \pi(\nu_2)$.

Now, $\mathrm{Hom}(W(G), G)$ is a semigroup, isomorphic to the semigroup G^G with the isomorphism π.

The group Σ_G is a group of all substitutions of the set G. Now it may be represented as a group of all invertible elements in the semigroup $\mathrm{Hom}(W(G), G)$. It is a subset in $\mathrm{Hom}(W(G), G)$ and an element in $\mathrm{Bool}(W(G), G)$.

Let X be an arbitrary finite set in X^0. Consider a mapping $\mu : X \to G$. This μ is simultaneously a mapping $\mu' : X \to W(G)$. We have homomorphisms $\overline{\mu} : W(X) \to G$ and $s : W(X) \to W(G)$ determined by μ and μ', respectively. Since s is, in fact, determined by the mapping $\mu : X \to G$, we write $s = s(\mu)$.

Apply these notations to the case when X coincides with G. Consider two mappings σ_1 and $\sigma_2 : G \to G$. We have $\overline{\sigma_1\sigma_2} = \overline{\sigma}_1\overline{\sigma}_2$. We are interested in the product $\overline{\sigma}_1 \cdot s(\sigma_2)$. It is easy to see that $\overline{\sigma}_1 \cdot s(\sigma_2) = \overline{\sigma_1\sigma_2}$.

Take further an arbitrary subalgebra R in $\mathrm{Hal}_\Theta(G)$, containing equalities. This means that R preserves operations of the type s^*, conjugated to the operations of the s_* type (compare [Pl1]).

We have $R(X) \subset \mathrm{Bool}(W(X), G)$ for every finite $X \subset X^0$. Since the algebra $R(X)$ is finite, it has atoms. Every element from $R(X)$ is a sum of atoms and all such atoms generate the algebra R.

Atoms are easily built. Let $\mu : W(X) \to G$ be a point in the space $\mathrm{Hom}(W(X), G)$. This space is a unit in the algebra $\mathrm{Bool}(W(X), G)$ and, simultaneously, a unit in $R(X)$. It contains the point μ.

Denote by (μ) the intersection of all elements of the algebra $R(X)$, containing the point μ. This (μ) is an atom in $R(X)$, and all atoms in $R(X)$ are built in such a way.

Consider further an algebra $R(G)$. It is a subalgebra in $\mathrm{Bool}(W(G), G)$. Take a unit e of the semigroup G^G. A point $\overline{e} : W(G) \to G$ which is a unit of the semigroup $\mathrm{Hom}(W(G), G)$ corresponds to this e. Take an atom $(\overline{e})$ in $R(G)$ determined by $\overline{e}$. Denote $(\overline{e}) = H = H^R$.

Proposition 16. *For every algebra R the subset H in $\mathrm{Hom}(W(G), G)$ is a subgroup in the group Σ_G.*

Proof. Let us make one general remark. Let a morphism $s : W(X) \to W(Y)$ be given. If $A \in R(X)$, then $s_*A \subset R(Y)$ for the given algebra R. Besides, we know that if $A \in R(Y)$, then $s^*A \in R(X)$.

Take now an arbitrary $\sigma : G \to G$. We have $\overline{\sigma} : W(G) \to G$ and $s = s(\sigma)$:

$W(G) \to W(G)$. Take, further, $s^* : \mathrm{Bool}(W(G), G) \to \mathrm{Bool}(W(G), G)$. By the construction, the atom H is an element in the algebra $\mathrm{Bool}(W(G), G)$. Take s^*H. It is also an element in $\mathrm{Bool}(W(G), G)$, consisting of all νs, $\nu \in H$. An element $\nu = \overline{e}$ is contained in H. Hence, $\overline{e}s(\sigma) = \overline{e\sigma} = \overline{\sigma} \in s^*H$.

Note also that both H and s^*H belong to the algebra $R(G)$. Assume that $\overline{\sigma} \in H$. Then $\overline{\sigma} \in H \cap s^*H$. This intersection is a nonzero element in $R(G)$. Since H is an atom, then $H \cap s^*H = H$, $H \subset s^*H$.

Assume that s^*H is strictly greater than H. Then nontrivial decomposition

$$s^*H = H \cup (\neg H \cap s^*H))$$

takes place. Apply a Boolean endomorphism s_*. Then

$$H \subset s_*(s^*H) = s_*H \cup s_*(\neg H \cap s^*H)).$$

This gives the decomposition

$$H = (H \cap s_*(H)) \cup (H \cap s_*(\neg H \cap s^*H)).$$

This decomposition is nontrivial (see [Pl1], p.347). Since the summands lie in $R(G)$ and H is an atom, the decomposition is impossible. Hence, $H = s^*H$. Let now $\overline{\sigma}_1$ and $\overline{\sigma}_2$ be two elements in H. Then $\overline{\sigma}_1 \cdot \overline{\sigma}_2 = \overline{\sigma_1\sigma_2} = \overline{s_1}s(\sigma_2) \in s^*H = H$, and the set H is closed under multiplication. Besides, $\overline{e} = \overline{\sigma}_1 s(\sigma) = \overline{\sigma_1\sigma}$, since $\overline{e} \in s^*H$. This means that for every $\overline{\sigma} \in H$ there is the inverse element $\overline{\sigma}_1 \in H$.

Hence, the whole set H is the subgroup in the group Σ_G.

Proposition 17. *Let X be an arbitrary finite set in X^0 and A be an atom of the Boolean algebra $R(X)$. Then there is a mapping $\mu : X \to G$ such that $s^*H = A$ holds for $s = s(\mu) : W(X) \to W(G)$.*

Proof. Take an arbitrary μ with $\overline{\mu} \in A$. Take $s = s(\mu)$ for such μ. Check that $s^*H = A$.

Take $\overline{e} \in H$ and $\overline{e}s(\mu) = \overline{e\mu} = \overline{\mu} \in A$. The same μ lies also in s^*H. We have $\overline{\mu} \in A \cap s^*H$. Both components lie in $R(X)$, and $A \subset s^*H$, since A is an atom. As previously, we see that $A = s^*H$.

Proposition 18. *The subgroup H in Σ_G coincides with the subgroup R' in* $\mathrm{Aut}(G)$.

Proof. We intend to prove that if $\sigma : G \to G$ is a substitution, then $\overline{\sigma} \in H$ if and only if σ is an automorphism of the algebra G, belonging to the subgroup R'.

Let, at the beginning, $\sigma \in R'$. Take the point $\overline{\sigma} : W(G) \to G$. It is easy to understand that $\sigma\overline{\mu} = \overline{\sigma\mu}$ for every mapping $\mu : X \to G$. If $X = G$, then $\sigma\overline{\mu} = \overline{\sigma\mu} = \overline{\sigma}\overline{\mu}$. In particular, $\sigma H = H$ for the given σ. This means that $\sigma\nu \in H \Leftrightarrow \nu \in H$. Take an element $\overline{e}$ in H. Then $\sigma\overline{e} = \overline{\sigma e} = \overline{\sigma} \in H$. Thus, $\sigma \in R'$ implies $\overline{\sigma} \in H$.

Let us prove the opposite. For every $A \subset \mathrm{Hom}(W(X), G)$ and $\sigma \in \Sigma_G$ define $\tilde{\sigma}A$. The set $\tilde{\sigma}A$ is the set of all $\mu : W(X) \to G$ for which $\tilde{\sigma}(\mu) = \nu \in A$. Here $\mu = \tilde{\sigma}^{-1}(\nu) \in \widetilde{\sigma^{-1}}{}^*A$. If $\sigma \in \mathrm{Aut}(G)$, then $\tilde{\sigma}A = \sigma A = (\sigma^{-1})^*A$.

Take now $A = H$ and let $\overline{\sigma}_1 \in H$. Check that $\widetilde{\sigma^{-1}}(\overline{\sigma}_1) = \overline{\sigma^{-1}}\;\overline{\sigma_1} = \overline{\sigma^{-1}\sigma_1}$. Take an arbitrary $g \in G$. Then

$$\widetilde{\sigma^{-1}}(\overline{\sigma}_1)(g) = \sigma^{-1}\sigma_1(g) = \overline{\sigma^{-1} \cdot \sigma_1}(g) = (\overline{\sigma^{-1}} \cdot \overline{\sigma}_1)(g).$$

Hence,

$$\widetilde{\sigma^{-1}}{}^*H = \overline{\sigma^{-1}}H.$$

In the right hand part there is a coset by H with the representative $\overline{\sigma^{-1}}$. In particular, if $\overline{\sigma} \in H$, then $\widetilde{\sigma^{-1}}{}^*H = H = \tilde{\sigma}H$.

Show now that $\tilde{\sigma}A = A$ for every atom A in $R(X)$. Take $\mu : X \to G$ with $\overline{\mu} \in A$ and let $s = s(\mu)$. Then $A = s^*H$, $s(x) \in G$ for every $x \in X$. We have

$$\tilde{\sigma}(A) = \widetilde{\sigma^{-1}}{}^*(A) = \widetilde{\sigma^{-1}}{}^*(s^*H) = s^*(\widetilde{\sigma^{-1}}{}^*H) = s^*(H) = A$$

for $\sigma \in \Sigma_G$. We used here the fact that $\widetilde{\sigma^{-1}}(\nu s) = \widetilde{\sigma^{-1}}(\nu) \cdot s$ holds for every $\nu \in H$. Thus, all atoms are invariant under every $\tilde{\sigma}$ with $\overline{\sigma} \in H$. But then all $A \in R$ are also invariant over such $\tilde{\sigma}$.

Show that this implies $\sigma \in \mathrm{Aut}(G)$.

Let $\omega \in \Omega$ be an n-ary operation, $a_1, \dots, a_n$ elements in G, $\overline{\sigma} \in H$. We need to check that $(a_1 \dots a_n\omega)^\sigma = a_1^\sigma \dots a_n^\sigma\omega$. Take variables $\{x_1, \dots, x_n, y\} \in X$ and proceed from the equality $x_1 \dots x_n\omega \equiv y$. This is the equality of the X type. Take further $\mu : W(X) \to G$ with the condition $\mu(x_1) = a_1, \dots, \mu(x_n) = a_n$ and $\mu(y) = b = a_1 \dots a_n\omega$. Then $\mu \in \mathrm{Val}(x_1 \dots x_n\omega = y) = A$. By the condition,

$A \in R(X)$, $\tilde{\sigma}(A) = A$. Thus, also $\tilde{\sigma}(\mu) \in A$. We have:

$$\begin{aligned}\tilde{\sigma}(\mu)(x_1 \dots x_n\omega) &= (\tilde{\sigma}(\mu)(x_1)) \dots (\tilde{\sigma}(\mu)(x_n))\omega \\ &= (\sigma\mu(x_1)) \dots (\sigma\mu(x_n))\omega = (\sigma a_1) \dots (\sigma a_n)\omega \\ &= \tilde{\sigma}(\mu)(y) = \sigma\mu(y) = \sigma b = \sigma(a_1 \dots a_n\omega).\end{aligned}$$

Hence, if $\tilde{\sigma} \in H$, then $\sigma \in \mathrm{Aut}(G)$ and $\tilde{\sigma}(A) = \sigma(A) = A$ for every $A \in R$, $\sigma \in R'$.

Let us finish the proof of Theorem 10.

Let R be a subalgebra in $\mathrm{Hal}_\Theta(G)$, containing equalities. We want to show that $R'' = R$. The inclusion $R \subset R''$ is always true. Take $H = R'$ and let $\overline{H}$ be a corresponding atom in $R(G)$. This $\overline{H}$ generates the whole algebra R. We have also $(R'')' = H$. Then the algebra R'' is generated by the same $\overline{H}$. Hence, $R'' = R$.

Let now H be a subgroup in $\mathrm{Aut}(G)$. A subset $\overline{H}$ in $\mathrm{Hom}(W(G), G)$, which is an element in $\mathrm{Bool}(W(G), G)$, corresponds to H. Let us generate a subalgebra R in $\mathrm{Hal}_\Theta(G)$ by this element. Check that $H = R' = \overline{H}'$. This implies that $H' = R'' = R$ and $H'' = R' = H$.

Let us pass to the proof of Theorem 12.

Let G_1 and G_2 be two finite algebras in the given variety Θ. Assume that the initial universum X^0 contains the sets G_1 and G_2 and consider Halmos algebras $\mathrm{Hal}_\Theta(G_1)$ and $\mathrm{Hal}_\Theta(G_2)$. We consider them together with the groups of automorphisms $\mathrm{Aut}(G_1)$ and $\mathrm{Aut}(G_2)$. Take subalgebras R_1 and R_2 in these algebras, respectively, both with equalities. Suppose that there is an isomorphism

$$\gamma : R_1 \to R_2.$$

We need to build an isomorphism $\delta : G_2 \to G_1$, inducing the given γ. Take a group $H_2 = R_2' \subset \mathrm{Aut}(G_2)$ by the given R_2. As before, take a set $\overline{H}_2 \subset \mathrm{Hom}(W(G_2), G_2)$ in $R_2(G_2)$. This set is an atom in $R_2(G_2)$ over the unit $\overline{e}$. Apply γ^{-1}. Then $A = \overline{H}_2^{\gamma^{-1}}$ is an atom in $R_1(G_2) = \mathrm{Bool}(W(G_2), G_1)$. Take an arbitrary $\mu : W(G_2) \to G_2$ in this nonempty set A. Let $\delta : G_2 \to G_1$ be the restriction of μ on G_2.

We want to show that every such δ solves the problem.

Consider a group $H_1 = R_1'$ in $\mathrm{Aut}(G_1)$, and let $\overline{H}_1$ be a corresponding atom in $R_1(G_1)$. We have also an atom A in $R_1(G_2)$. As we know, $A = s^*\overline{H}_1 = H_2^{\gamma^{-1}}$.

Take $B = \overline{H}_1^{\gamma} = s_1^*(\overline{H}_2)$ by $\overline{H}_1$. Here $s_1 = s(\sigma)$ with $\sigma : G_1 \to G_2$, selected in B as δ in A.

Using ([Pl1], p.351), we prove that $\delta : G_2 \to G_1$ is a bijection, as well as $\sigma : G_1 \to G_2$. We also prove that δ and σ are isomorphisms of algebras ([Pl1], p.356) and that $H_1 = \delta H_2 \delta^{-1}$, or, similarly $R'_1 = \delta R'_2 \delta^{-1}$. This allows to show that $\delta_* : \mathrm{Hal}_\Theta(G_1) \to \mathrm{Hal}_\Theta(G_2)$ induces an isomorphism $R_1 \to R_2$. We check further that this isomorphism coincides with the initial γ. It is sufficient to verify the last one on the set A, generating the algebra R_1. The equality $\gamma(A) = \delta_*(A)$ follows from the definition.

7. Geometric properties of models

7.1. Isomorphisms of categories.

We consider a new model (G_1, Φ, f^δ) for the given isomorphism $\delta : G_2 \to G_1$ and a model (G_1, Φ, f). This new model is isomorphic to the initial one. We consider also the commutative diagram

$$\begin{tikzcd} \mathrm{Hal}_\Theta(\Phi) \arrow[r, "\mathrm{Val}_f"] \arrow[dr, "\mathrm{Val}_{f^\delta}"'] & \mathrm{Hal}_\Theta(G_1) \arrow[d, "\delta_*"] \\ & \mathrm{Hal}_\Theta(G_2) \end{tikzcd}$$

If $A = T^f$ is an algebraic set in $\mathrm{Hal}_\Theta(G_1)$, then $\delta_* T^{f^\delta}$ is an algebraic set in $\mathrm{Hal}_\Theta(G_2)$. Hence, we have a bijection between f-algebraic sets in $\mathrm{Hal}_\Theta(G_1)$ and f^δ-algebraic sets in $\mathrm{Hal}_\Theta(G_2)$. We can speak now of a commutative diagram

$$\begin{tikzcd} \mathrm{Hal}_\Theta(\Phi) \arrow[r, "\mathrm{Val}_f"] \arrow[dr, "\mathrm{Val}_{f^\delta}"'] & K_{\Phi\Theta}(f) \arrow[d, "\delta_*"] \\ & K_{\Phi\Theta}(f^\delta) \end{tikzcd}$$

Here the bijection δ_* is well coordinated with the morphisms of the categories, and thus, is an isomorphism of categories. Besides, δ_* induces an isomorphism of algebras R_f and R_{f^δ}.

We are interested in the general problem of isomorphism of two categories of the type $K_{\Phi\Theta}(f)$. Here we define a *special* isomorphism. For the models (G_1, Φ, f_1) and (G_2, Φ, f_2), it is an isomorphism, determined by the diagram

$$\begin{tikzcd} \mathrm{Hal}_\Theta(\Phi) \arrow[r, "\mathrm{Val}_{f_1}"] \arrow[dr, "\mathrm{Val}_{f_2}"'] & K_{\Phi\Theta}(f_1) \arrow[d, "\gamma"] \\ & K_{\Phi\Theta}(f_2) \end{tikzcd}$$

Here γ is an isomorphism of categories. It follows from the definition that the same γ induces an isomorphism of algebras R_{f_1} and R_{f_2}. Besides, $\gamma(T^{f_1}) = T^{f_2}$ for every $T \subset \mathrm{Hal}_\Theta(\Phi)$ of the definite type X.

Theorem 13. *If the algebras G_1 and G_2 are finite, then the categories $K_{\Phi\Theta}(f_1)$ and $K_{\Phi\Theta}(f_2)$ are specialy isomorphic if and only if the models are isomorphic.*

Proof. We have already seen this in one direction. Now let us have a special isomorphism $\gamma : K_{\Phi\Theta}(f_1) \to K_{\Phi\Theta}(f_2)$. We show that there exists an isomorphism of models $\delta : G_2 \to G_1, f_1 = f_2^\delta$, such that $\gamma = \delta_*$ on the objects of the categories. We have an isomorphism $\gamma : R_{f_1} \to R_{f_2}$. According to Theorem 12, there is an isomorphism of algebras $\delta : G_2 \to G_1$, such that γ and δ_* coincide on the elements from R_{f_1}. From this follows that γ and δ_* coincide on the objects of the categories. It is also easy to see that the commutative diagram

$$\begin{tikzcd} \mathrm{Hal}_\Theta(\Phi) \arrow[r, "\mathrm{Val}_{f_1}"] \arrow[dr, "\mathrm{Val}_{f_2}"'] & \mathrm{Hal}_\Theta(G_1) \arrow[d, "\delta_*"] \\ & \mathrm{Hal}_\Theta(G_2) \end{tikzcd}$$

takes place.

Applying the remarks from the end of the Section 7, we conclude that $f_1 = f_2^\delta$ and the models are isomorphic.

Along with the special isomorphisms we consider *strict* isomorphisms. It is an isomorphism $\gamma : K_{\Phi_1\Theta}(f_1) \to K_{\Phi_2\Theta}(f_2)$, inducing an isomorphism of algebras $R_{f_1} \to R_{f_2}$. Here Φ_1 and Φ_2 are different in general, and the models have the form (G_1, Φ_1, f_1) and (G_2, Φ_2, f_2). We do not assume relations with the algebras of formulas. We cannot speak here about isomorphism of models and we use the notion of automorphic isomorphism.

Definition. *The modes (G_1, Φ_1, f_1) and (G_2, Φ_2, f_2) are called automorphically isomorphic, if*

1. Algebras G_1 and G_2 are isomorphic.

2. There exists an isomorphism of algebras $\delta : G_2 \to G_1$ with $\mathrm{Aut}(f_1) = \delta \mathrm{Aut}(f_2)\delta^{-1}$.

Here the groups of automorphisms of models are conjugated.

If the models are, in particular, isomorphic, $\Phi_1 = \Phi_2$, then they are automorphically isomorphic.

Theorem 14. *For the finite algebras G_1 and G_2 the categories $K_{\Phi_1\Theta}(f_1)$ and $K_{\Phi_1\Theta}(f_2)$ are strictly isomorphic if and ony if the models are automorphically isomorphic.*

Proof. Let a strict isomorphism

$$\gamma : K_{\Phi_1\Theta}(f_1) \to K_{\Phi_2\Theta}(f_2)$$

be given. By the condition, this γ induces an isomorphism of algebras $\gamma : R_{f_1} \to R_{f_2}$. Once more by Theorem 12, we have an isomorphism $\delta : G_2 \to G_1$ such that δ_* induces γ. Here, $R_{f_2} = \delta_*(R_{f_1})$. This, in its turn, gives $R'_{f_1} = \delta R'_{f_2}\delta^{-1}$. But we know from Theorem 9 that $R'_{f_1} = \mathrm{Aut}(f_2)$, $R'_{f_2} = \mathrm{Aut}(f_2)$. Thus, $\mathrm{Aut}(f_1)$ and $\mathrm{Aut}(f_2)$ are conjugated by the isomorphism $\delta : G_2 \to G_1$. We have verified that the models are automorphically isomorphic.

Let us prove the opposite. Let the models (G_1, Φ_1, f_1) and (G_2, Φ_2, f_2) be automorphically isomorphic with the isomorphism of algebras $\delta : G_2 \to G_1$. We have an isomorphism $\delta_* : \mathrm{Hal}_\Theta(G_1) \to \mathrm{Hal}_\Theta(G_2)$. We have a subalgebra R_{f_1} in $\mathrm{Hal}_\Theta(G_1)$ and R_{f_2} in $\mathrm{Hal}_\Theta(G_2)$. Here $R'_{f_1} = \mathrm{Aut}(f_1)$ and $R'_{f_2} = \mathrm{Aut}(f_2)$. By the condition we have $R'_{f_1} = \delta R'_{f_2}\delta^{-1}$. This means that $\delta_*(R_{f_1}) = R_{f_2}$. Indeed, $(\delta_*(R_{f_1}))' = \delta^{-1}R'_{f_1}\delta = R'_{f_2}$. Hence, $\delta_*(R_{f_1}) = R_{f_2}$ and $\delta_*(R)' = \delta^{-1}R'\delta$. This always takes place. Then δ_* induces an isomorphism $R_{f_1} \to R_{f_2}$.

Let now $A \in R_{f_1}$. Then $A = \mathrm{Val}_{f_1}(u)$ for some $u \in \mathrm{Hal}_\Theta(\Phi_1)$, $\delta_*(A) = B \in R_{f_2}$, $B = \mathrm{Val}_{f_2}(v)$, $v \in \mathrm{Hal}_\Theta(\Phi_2)$. Here u and v are of the same type X. We have $\delta_*\mathrm{Val}_{f_1}(u) = \mathrm{Val}_{f_2}(v)$.

Let, further, A be an object of the category $K_{\Phi_1\Theta}(f_1)$, $A = T^{f_1}$, where T is a collection of formulas in $\mathrm{Hal}_\Theta(\Phi)$ and T is of type X. We have

$$A = \bigcap_{u\in T} \mathrm{Val}_{f_1}(u).$$

We have also

$$\delta_*(A) = \bigcap_{u\in T} \delta_*\mathrm{Val}_{f_1}(u) = \bigcap_{v\in T^*} \mathrm{Val}_{f_2}(v).$$

Here T^* is a collection of formulas v in $\mathrm{Hal}_\Theta(\Phi_2)$ of the type X, somehow connected with the collection T. In any case, we may claim that δ_*A, as well as A, is an

algebraic set. This means that there exists a bijection on the objects $K_{\Phi_1\Theta}(f_1) \to K_{\Phi_2\Theta}(f_2)$. Let now $s : A_1 \to A_2$ be a morphism in $K_{\Phi,\Theta}(f_1)$. We have $A_1 \subset sA_2$. But then $\delta_*(A_1) \subset \delta_*(sA_2) = s\delta_* A_2$. Hence, the same s gives a morphism $s : \delta_*(A_1) \to \delta_*(A_2)$. The opposite is also true. Thus, we come to the isomorphism of categories $\delta_* : K_{\Phi,\Theta}(f_1) \to K_{\Phi_2\Theta}(f_2)$, and this isomorphism is a strict one. This completes the proof of Theorem 14.

7.2. A remark on the categories $K_{\Phi\Theta}(G)$.

Here G is an arbitrary algebra in Θ. Recall that the objects of this category have the form (X, A, f), where f is a interpretation of the given set Φ in the given algebra G.

Consider $K_{\Phi\Theta}(G_1)$ and $K_{\Phi\Theta}(G_2)$, while the isomorphism $\delta : G_1 \to G_2$ is given. To the model (G_1, Φ, f) corresponds a model (G_2, Φ, f^δ). Simultaneously, to the object (X, A, f) corresponds an object $(X, \delta_*(A), f^\delta)$. This gives an isomorphism of categories

$$K_{\Phi\Theta}(G_1) \to K_{\Phi\Theta}(G_2).$$

If, further, $\delta : G \to G$ is an automorphism of algebras, then an automorphism of categories $K_{\Phi\Theta}(G)$ corresponds to δ. This leads to the representation

$$\mathrm{Aut}(G) \to \mathrm{Aut}(K_{\Phi\Theta}(G)).$$

This representation is naturally tied with the representation

$$\mathrm{Aut}(G) \to \mathrm{Aut}(\mathrm{Hal}_\Theta(G)).$$

7.3. Geometrical properties .

Definition 1. *The model (G, Φ, f) is called geometrically noetherian if for every finite set X and every set of formulas T in $\mathrm{Hal}_{\Phi\Theta}(X)$ there is some finite part $T_0 \subset T$ with $T_0^f = T^f$. This means that $T_0^{ff} = T^{ff}$.*

Theorem 15. *The model (G, Φ, f) is geometrically noetherian if and only if the minimality condition holds in the lattice $\mathrm{Alv}_f(X)$. Correspondingly, in the lattice $\mathrm{Cl}_f(X)$ we have the maximality condition.*

Now let $T_0 = \{u_1, \ldots, u_n\}$ and u is $u_1 \wedge \cdots \wedge u_n$. Then $T_0^f = \mathrm{Val}_f(u) = \{u\}^f$. Thus, we may claim that if the model (G, Φ, f) is geometrically noetherian, then every algebraic set over this model is a simple algebraic set.

However, the corresponding element $u = u_1 \wedge \cdots \wedge u_n$ does not necessarily belong to the initial set T. We call a model (G, Φ, f) *weak geometrically noetherian* if every algebraic set over it is a simple algebraic set. Weak noetherian model is not necessarily geometrically noetherian.

However, we may claim that every finite model is geometrically noetherian (compare Theorem 8 in Galois theory). We may also claim that any finite cartesian product of geometrically noetherian models is also a geometrically noetherian model. A submodel of a geometrically noetherian model is also geometrically noetherian. The similar properties are not true in general, in respect to to cartesian powers.

Let us pass to the notion of geometrical equivalence of two models. Let two models (G_1, Φ, f_1) and (G_2, Φ, f_2) with the same Φ be given.

Definition 2. *The models (G_1, Φ, f_1) and (G_2, Φ, f_2) are geometrically equivalent if $T^{f_1 f_1} = T^{f_2 f_2}$ holds for every finite X and every T in $\mathrm{Hal}_{\Phi\Theta}(X)$.*

If the models (G_1, Φ, f_1) and (G_2, Φ, f_2) are geometrically equivalent, then

1. The lattices $\mathrm{Cl}_{f_1}(X)$ and $\mathrm{Cl}_{f_2}(X)$ coincide, while the lattices $\mathrm{Alv}_{f_1}(X)$ and $\mathrm{Alv}_{f_2}(X)$ are isomorphic.

2. The categories $C_{\Phi\Theta}(f_1)$ and $C_{\Phi\Theta}(f_2)$ coincide, while the categories $K_{\Phi\Theta}(f_1)$ and $K_{\Phi\Theta}(f_2)$ are isomorphic.

3. These models are elementary equivalent.

It follows from the first claim that if the models are geometrically equivalent and one of them is geometrically noetherian, then the second one is also geometrically noetherian.

Theorem 16. *Let the model (G, Φ, f) be geometrically noetherian. Then each of its utrapower is also geometrically noetherian, and all these ultrapowers are geometrically equivalent to the initial (G, Φ, f).*

All the described notions are naturally tied with the logic of generalized (infinitary) formulas of the kind $(\wedge_{u \in T}) \to v$, or, what is the same $T \to v$. For the geometrically noetherian models it is sufficient to proceed from the usual finite formulas.

Elementary equivalence of the models does not generally imply their geometrical equivalence. However, we may claim the following

Theorem 17. *If two models are elementary equivalent and one of them is geometrically noetherian, then the second one is also geometrically noetherian and these models are geometrically equivalent.*

In concern with the notion of geometrically noetherian model let us return to the notion of the logical kernel of a homomorphism of the form $\mu : W(X) \to G$. It is easy to see that if $\mathrm{Log}_f\mathrm{Ker}(\mu)$ is such a kernel and $\mathrm{Val}_f(\mathrm{Log}_f\mathrm{Ker}(\mu))$ is its image in the algebra $R_f(X)$ then this image is a principal ultrafilter if the model (G, Φ, f) is geometrically noetherian. This ultrafilter is generated by the algebraic set $\{\mu\}^{ff}$.

8. Applications to the knowledge science

8.1 Introduction.

Knowledge theory and knowledge bases provide an important example of the field where applications of universal algebra and algebraic logic are very natural, and their interacting with quite practical problems arising in computer science is very productive. Another examples of such interaction are given by relational database theory, constraint satisfaction problem ([BJ],[JCP]), theory of complexity, and by others.

One can speak about knowledge and a system of knowledge. As a rule, a domain of knowledge or of a system of knowledge is fixed. In our approach only knowledge that allows a formalization in some logic is considered. The logic may be different. It is often oriented towards the corresponding field of knowledge cf. [G],[L],[S].

In this paper we focus on the special situation of elementary knowledge.

Elementary knowledge is considered to be a first order knowledge, i.e., the knowledge that can be represented by the means of the First Order Logic (FOL). The corresponding applied field (field of knowledge) is grounded on some variety of algebras Θ, which is arbitrary but fixed. This variety Θ is considered as a knowledge type. Its counterpart in database theory is the notion of datatype Θ.

We also fix a set of symbols of relations Φ. The subject of knowledge is a triple (G, Φ, f), where G is an algebra in Θ and f is a interpretation of the set Φ in G. It is a model in the ordinary mathematical sense. As a rule, we use shorthand and write f instead of (G, Φ, f). For the given Φ we denote the corresponding applied field by $\Phi\Theta$.

FOL is also oriented towards the variety Θ.

We assume that every knowledge under consideration is represented by three components:

1) *The description of knowledge.* It is a syntactical part of knowledge, written out in the language of the given logic. The description reflects, what do we want to know.

2) *The subject of knowledge* which is an object in the given applied field, i.e., an object for which we determine knowledge.

3) *The content of knowledge* (its semantics).

The first two components are relatively independent, while the third one is uniquely determined by the previous two. In the theory under consideration, this third component has a geometrical nature. In some sense it is an algebraic set in an affine space. If T is a description of knowledge and (G, Φ, f) is a subject, then T^f denotes the content of knowledge. We would like to equip the content with its own structure, algebraic or geometric, and to consider some aspects of such structure.

We want to underline that there are three aspects in our approach to knowledge representation: logical (for knowledge description), algebraic (for the subject of knowledge) and geometric (in the content of knowledge). This geometry is of algebraic nature. However, the involved algebra inherits some geometric intuition.

We consider categories of elementary knowledge. The language of categories in knowledge theory is a good way to organize and systematize primary elementary knowledge. Morphisms in a knowledge category give links between knowledge. In particular, one can speak of isomorphic knowledge. The categorical approach also allows us to use ideas of monada and comonada [ML]. It turns out that this leads to some general views on enrichment and computation of knowledge. Enrichment of a structure can be associated with a suitable monada over a category, while the corresponding computation is organized by comonada.

Let us make one more remark. In every well described field of knowledge, one can study a category of elementary knowledge belonging to this field. Consideration of such categories might be of special interest.

8.2 The category of knowledge.

Define first the category $\mathrm{Know}_{\Phi\Theta}(f)$.

Fix a model (subject of knowledge) (G, Φ, f). Let us define a category of knowledge for this model and denote it by $\mathrm{Know}_{\Phi\Theta}(f)$. This is the knowledge category for the given subject of knowledge. Since the model is fixed, the objects of the category Knowf have to have the form (X, T, A). We do not fix the subject of knowledge in the notation of the object, since it is fixed in the notation of the category.

The set X is multi-sorted. It marks the "place" where the knowledge is situated. The set X points also the "place of the knowledge", i.e., the space of the knowledge $Hom(W(X), G)$, while the subject of the knowledge (G, Φ, f) is given. The set T is the description of the knowledge in the algebra $Hal_\Theta(X)$, and $A = T^f$ is the content of knowledge, depending on T and f. The set $T^{ff} = A^f$ is the full description of the knowledge (X, T, A) which is a Boolean filter in $\mathrm{Hal}_{\Phi\Theta}(X)$.

Now about morphisms $(X, T_1, A) \to (Y, T_2, B)$. Take $s\colon W(Y) \to W(X)$. We have also $s\colon \mathrm{Hal}_{\Phi\Theta}(Y) \to \mathrm{Hal}_{\Phi\Theta}(X)$ (see 2.2). This is a homomorphism of Boolean algebras. The homomorphism s gives rise to

$$\tilde{s}\colon Hom(W(X), G) \to Hom(W(Y), G).$$

As above, the first s is admissible for A and B if $\tilde{s}(\nu) = \nu s \in B$ for every point $\nu\colon W(X) \to G$ in A.

As we know, s is admissible for A and B if and only if $su \in A^f$ for every $u \in B^f$. This holds for s_*, for which we have also a homomorphism $\overline{s} : \mathrm{Hal}_{\Phi\Theta}(Y)/B^f \to \mathrm{Hal}_{\Phi\Theta}(X)/A^f$. It is easy to prove that s is admissible for A and B if and only if $su \in A^f$ holds for every $u \in T_2$. We consider admissible s as a morphism

$$s\colon (X, T_1, A) \to (Y, T_2, B),$$

in the weak category Knowf.

We have $\tilde{s}(\nu) = \nu s \in B$ if $\nu \in A$, and s induces a mapping $[s]\colon A \to B$. Simultaneously, there is a mapping $s\colon T_2 \to A^f$ and a homomorphism

$$\overline{s}\colon \mathrm{Hal}_{\Phi\Theta}(Y)/B^f \to \mathrm{Hal}_{\Phi\Theta}(X)/A^f.$$

We have already mentioned (Proposition 2) that $\overline{s}_1 = \overline{s}_2$ follows from $[s_1] = [s_2]$. Thus, we can take the morphisms of the form

$$[s]\colon (X, T_1, A) \to (Y, T_2, B),$$

for the morphisms of the exact category Knowf. The canonical functors Knowf$\to$ $K_{\Phi\Theta}(f)$ for weak and exact categories are given by the transition $(X, T, A) \to (X, A)$. In this transition we "forget" to fix the description of knowledge T.

Now we define the category $\mathrm{Know}_{\Phi\Theta}$. Let us define the category of elementary knowledge for the whole applied field $\Phi\Theta$; the subject of the knowledge (G, Φ, f) is not fixed. As earlier, we proceed from the category $\Phi\Theta$ whose morphisms are homomorphisms in Θ. They ignore the relations from Φ.

An object of the knowledge category Know has the form

$$(X, T, A; (G, \Phi, f)),$$

and we write $(X, T, A; G, f)$, because Φ is fixed for the category. Here X marks the place of knowledge. The components $A = T^f$, G and f may change.

Consider morphisms:

$$(X, T_1, A; G_1, f_1) \to (Y, T_2, B; G_2, f_2).$$

We apply the same approach as in Section 3.3 with some modifications.

Start from $s : W(Y) \to W(X)$ and $\delta : G_1 \to G_2$. These s and δ should correlate. Let us explain the correlation condition. Take a set $A_1 = \{\delta\nu, \nu \in A\} = \delta^* A$ and take further $T_1^\delta = A_1^{f_2}$. Correlation of s and δ means that $su \in T_1^\delta$ holds for any $u \in T_2$. The same holds for every $u \in B^{f_2}$. The last also says that there is a homomorphism

$$\overline{s}\colon \mathrm{Hal}_{\Phi\Theta}(Y)/B^{f_2} \to \mathrm{Hal}_{\Phi\Theta}(X)/A_1^{f_2}.$$

The first of the two mappings $(s, \delta)\colon A \to B$ and $s\colon T_2 \to T_1^\delta$ transforms the content of knowledge, while the second one acts on the description. Here T_2 and T_1^δ describe knowledge associated with the same subject (G_2, Φ, f_2).

With the fixed δ there is also an exact mapping $([s], \delta) : A \to B$. This brings us to weak and exact categories Know. The morphisms of the first one are (s, δ) and in the second one they are of the form $([s], \delta)$ for $(X, T_1, A; G_2, f_1) \to (Y, T_2, B; G_2, f_2)$. The canonical functors Know$\to$ $K_{\Phi\Theta}$ are defined by the transition

$$(X, T, A; G, f) \to (X, A; G, f).$$

As above, we remove the description of knowledge from the notations.

For given algebras G in Θ we consider the categories $K_{\Phi\Theta}(G)$ and $\mathrm{Know}_{\Phi\Theta}(G)$.

An algebra $G \in \Theta$ is fixed in the categories $K_{\Phi\Theta}(G)$ and $\mathrm{Know}_{\Phi\Theta}(G)$. A set of symbols of relations Φ is fixed as usual, but interpretations f of Φ in G may change. Thus, $K_{\Phi\Theta}(G)$ is a subcategory in $K_{\Phi\Theta}$ and $\mathrm{Know}_{\Phi\Theta}(G)$ is a subcategory in $\mathrm{Know}_{\Phi\Theta}$. Here the corresponding $\delta : G \to G$ are identical homomorphisms. Objects of the category $K_{\Phi\Theta}(G)$ have the form (X, A, f), and those of the category $\mathrm{Know}_{\Phi\Theta}(G)$ are written as (X, T, A, f). There is a canonical functor $\mathrm{Know}_{\Phi\Theta}(G) \to K_{\Phi\Theta}(G)$. As for morphisms

$$(X, A, f_1) \to (Y, B, f_2) \text{ and}$$

$$(X, T_1, A, f_1) \to (Y, T_2, B, f_2),$$

we note that $A = A_1, A_1^{f_2} = T_1^{\delta}$ and $A^{f_2} = T_1^{f_1 f_2}$. Hence, the corresponding admissible $s : W(Y) \to W(X)$ transfers each $u \in T_2$ into $su \in T_1^{f_1 f_2}$ and it induces a homomorphism

$$\overline{s} : \mathrm{Hal}_{\Phi\Theta}(Y)/B^{f_2} \to \mathrm{Hal}_{\Phi\Theta}(X)/A^{f_2}.$$

Every s yields a mapping $[s] : A \to B$. This provides a morphism $(X, A, f_1) \to (Y, B, f_2)$.

8.3 Category of knowledge description (the category $L_\Theta(\Phi)$). .

Denote the category of knowledge description by $L_{\Phi\Theta}$ or $L_\Theta(\Phi)$.

Its objects are of the form (X, T), where X is a finite set and T is a set of formulas of $\mathrm{Hal}_{\Phi\Theta}(X)$. Define morphisms $(X, T_1) \to (X, T_2)$. According to description of the category $\mathrm{Hal}_\Theta(\Phi)$ proceed from the functor $\Theta^0 \to \mathrm{Hal}_\Theta(\Phi)$ which assigns a mapping $s_* : \mathrm{Hal}_{\Phi\Theta}(X) \to \mathrm{Hal}_{\Phi\Theta}(Y)$ to every homomorphism $s : W(X) \to W(Y)$. We say that s is*admissible in respect to* T_1 *and* T_2 if $s_*(u) \in T_2$ for every $u \in T_1$. For such admissible s we have a mapping $s_* : T_1 \to T_2$ which determines

$$s_* : (X, T_1) \to (X, T_2)$$

8.4 Functor of transition from knowledge description to knowledge content (the functor Ct_f).

Proceed from the model (G, Φ, f) and consider a functor

$$\mathrm{Ct}_f : L_{\Phi\Theta} \to K_{\Phi\Theta}(f).$$

Here, $K_{\Phi\Theta}(f)$ is the corresponding category of algebraic (elementary) sets over the given model and Ct stands for "contents". The functor Ct_f is a contravariant one. To every object (X,T) of the category $L_{\Phi\Theta}$ it assigns the corresponding content $(X,T^f)=(X,A)$ which is an object of the category $K_{\Phi\Theta}(f)$.

Now one has to define the functor Ct_f on morphisms. Let a morphism

$$s_*:(Y,T_2)\to(X,T_1)$$

be given for $s:W(Y)\to W(X)$. Show that s induces a morphism

$$\widetilde{s_*}:(X,A)\to(Y,B),$$

where $A=T_1^f$, and $B=T_2^f$.

We proceed from $\widetilde{s}:\mathrm{Hom}(W(X),G)\to\mathrm{Hom}(W(Y),G)$.

Let us define a transition $s\to\widetilde{s}$.

Check first that if s is admissible for T_2 and T_1 then this s is admissible for $A=T_1^f$ and $B=T_2^f$. The last means that $\widetilde{s}(\nu)\in B$ if $\nu\in A$. The inclusion $\nu\in A$ says that $\nu\in\mathrm{Val}_f(v)$ for every $v\in T_1$. We need to verify that $\nu s\in B$, that is $\nu s\in\mathrm{Val}_f(u)$ for every $u\in T_2$.

Take an arbitrary $u\in T_2$. We have: $v=s_*(u)\in T_1$; $\nu\in\mathrm{Val}_f(v)=\mathrm{Val}_f(s_*u)=s\mathrm{Val}_f(u)$. This gives $\nu s\in\mathrm{Val}_f(u)$. We used that s and Val_f commute, since Val_f is a homomorphism of algebras.

The mapping $[s]:A\to B$ corresponds to the homomorphism $s:W(Y)\to W(X)$. This mapping is considered simultaneously as a morphism in the category $K_{\Phi\Theta}(f)$ (see 3.2)

$$[s]:(X,A)\to(Y,B).$$

We define: $\mathrm{Ct}_f(s_*)=\widetilde{s}_*=[s]$.

Check now compatibility of the definition of Ct_f with the multiplication of morphisms. Given $s_1:W(X)\to W(Y)$ and $s_2:W(Y)\to W(Z)$ we have $s_2s_1:W(X)\to W(Z)$. Using the fact that the transition $\Theta^0\to\mathrm{Hal}_\Theta(\Phi)$ is a functor, we get $(s_2s_1)_*=s_{2*}s_{1*}$. Here, we have

$$s_{1*}:\mathrm{Hal}_{\Phi\Theta}(X)\to\mathrm{Hal}_{\Phi\Theta}(Y),$$

$$s_{2*}:\mathrm{Hal}_{\Phi\Theta}(Y)\to\mathrm{Hal}_{\Phi\Theta}(Z),$$

and

$$(s_2 s_1)_* : \mathrm{Hal}_{\Phi\Theta}(X) \to \mathrm{Hal}_{\Phi\Theta}(Z).$$

Let (X, T_1), (Y, T_2) and (Z, T_3) be objects in $L_\Theta(\Phi)$, and s_1, s_2 admissible in respect to T_1, T_2 and, correspondingly, for T_2, T_3. In this case there are morphisms

$$s_{1*} : (X, T_1) \to (Y, T_2),$$

$$s_{2*} : (Y, T_2) \to (Z, T_3),$$

and

$$s_{2*} s_{1*} = (s_2 s_1)_* : (X, T_1) \to (Z, T_3).$$

Take $T_1^f = A$, $T_2^f = B$, $T_3^f = C$. We have

$$\widetilde{s_{1*}} : (Y, B) \to (X, A),$$

$$\widetilde{s_{2*}} : (Z, C) \to (Y, B),$$

and

$$\widetilde{s_2 s_{1*}} = \widetilde{s_{1*}} \widetilde{s_{2*}} : (Z, C) \to (X, A).$$

This gives compatibility of the functor Ct_f with the multiplication of morphisms. Compatibility with the unity morphism is evident. This finishes the definition of the contravariant functor $\mathrm{Ct}_f : L_{\Phi\Theta} \to K_{\Phi\Theta}(f)$.

8.5 Homomorphisms of Halmos algebras $\mathrm{Hal}_\Theta(\Phi)$ and functors of the categories $L_\Theta(\Phi)$.

Given a homomorphism $\beta : \mathrm{Hal}_\Theta(\Phi_1) \to \mathrm{Hal}_\Theta(\Phi_2)$, define the corresponding functor $\widetilde{\beta} : L_\Theta(\Phi_1) \to L_\Theta(\Phi_2)$. For every set of formulas $T \subset \mathrm{Hal}_{\Phi_1\Theta}(X)$, denote by T^β the set $T^\beta = \{u^\beta, u \in T\}$. If (X, T) is an object in $L_\Theta(\Phi_1)$, then, setting

$$\widetilde{\beta}(X, T) = (X, T^\beta),$$

we get an object in $L_\Theta(\Phi_2)$.

In order to define the functor $\widetilde{\beta}$ on morphisms let us make a remark. Proceed from the functors $\Theta^0 \to \mathrm{Hal}_\Theta(\Phi_1)$ and $\Theta^0 \to \mathrm{Hal}_\Theta(\Phi_2)$. The morphisms

$$s_*^1 : \mathrm{Hal}_{\Phi_1\Theta}(X) \to \mathrm{Hal}_{\Phi_1\Theta}(Y),$$

$$s^2_* : \mathrm{Hal}_{\Phi_2\Theta}(X) \to \mathrm{Hal}_{\Phi_2\Theta}(Y)$$

correspond to every $s : W(X) \to W(Y)$. We have also

$$\beta = (\beta_X, X \in \Gamma^0) : \mathrm{Hal}_\Theta(\Phi_1) \to \mathrm{Hal}_\Theta(\Phi_2).$$

The fact that the homomorphism β is compatible with the operation s is represented by the commutative diagram

$$\begin{tikzcd} \mathrm{Hal}_{\Phi_1\Theta}(X) \arrow[r, "s^1_*"] \arrow[d, "\beta_X"'] & \mathrm{Hal}_{\Phi_1\Theta}(Y) \arrow[d, "\beta_Y"] \\ \mathrm{Hal}_{\Phi_2\Theta}(X) \arrow[r, "s^2_*"] & \mathrm{Hal}_{\Phi_2\Theta}(Y) \end{tikzcd}$$

So, for a homomorphism $s : W(X) \to W(Y)$ we have the equality $\beta_Y s^1_*(u) = s^2_*\beta_X(u)$ for every $u \in \mathrm{Hal}_{\Phi_1\Theta}(X)$.

Now we are able to define an action of the functor $\widetilde{\beta}$ on morphisms. Let a morphism $s^1_* : (X, T_1) \to (Y, T_2)$ in the category $L_{\Phi_1\Theta}$ be given and $s^1_*(u) \in T_2$ if $u \in T_1$. Then, we have $s^2_*(v) \in T_2^\beta$ if $v \in T_1^\beta$.

Indeed, let $v = \beta_X(u)$, $u \in T_1$, $v \in T_1^{\beta_X}$. We have:

$$s^2_*\beta_X(u) = s^2_*(v) = \beta_Y s^1_*(u) \in T_2^{\beta_Y},$$

since $s^1_*(u) \in T_2$. Hence, $s^2_*(v) \in T_2^{\beta_Y}$ for every $v = \beta_X(u) \in T_1^{\beta_X}$.

We set $s^2_* = \widetilde{\beta}(s^1_*) : T_1^{\beta_X} \to T_2^{\beta_Y}$. A morphism

$$s^2_* = \widetilde{\beta}(s^1_*) : (X, T_1^{\beta_X}) \to (Y, T_2^{\beta_Y})$$

corresponds to $s^1_* : (X, T_1) \to (Y, T_2)$.

Check now compatibility of the transition $s^1_* \to s^2_*$ with the multiplication of morphisms. Given $s_1 : W(X) \to W(Y)$ and $s_2 : W(Y) \to W(Z)$, we have $s_2 s_1 : W(X) \to W(Z)$. Using once more the fact that the transition $\Theta^0 \to \mathrm{Hal}_\Theta(\Phi)$ is a functor, we get

$$(s^1_2 s^1_1)_* = s^1_{2*} s^1_{1*},$$

$$(s^2_2 s^2_1)_* = s^2_{2*} s^2_{1*},$$

Apply $\widetilde{\beta}$. We need to verify that $\widetilde{\beta}(s^1_{2*}s^1_{1*}) = \widetilde{\beta}(s^1_{2*})\widetilde{\beta}(s^1_{1*})$. We have

$$\widetilde{\beta}(s^1_{2*}s^1_{1*}) = \widetilde{\beta}(s^1_2 s^1_1)_* = (s^2_2 s^2_1)_* = s^2_{2*}s^2_{1*} = \widetilde{\beta}(s^1_{2*})\widetilde{\beta}(s^1_{1*}).$$

This gives compatibility with the multiplication as well as with the unit. Hence, we have the functor $\widetilde{\beta} : L_\Theta(\Phi_1) \to L_\Theta(\Phi_2)$.

8.6 Knowledge bases.

We proceed from a multi-model (G, Φ, F). A multi-model (G, Φ, F) defines a system of models $(G, \Phi, f,)$ where f runs the set F. Here G is an algebra in Θ, and Φ is a set of relations. Recall that both the algebra $G \in \Theta$ and a relation $f \in F$ are multi-sorted. The set F is a set of instances f, where f is a interpretation of the set Φ in G.

To every such multi-model corresponds a knowledge base $KB = KB(G, \Phi, F)$. The definition slightly differs from that of [PTP].

Definition. *A knowledge base $KB = KB(G, \Phi, F)$ consists of two categories. The first one is the category of knowledge description $L_\Theta(\Phi)$, and the second one is the category of knowledge content $K_{\Phi\Theta}(f)$. These two categories are related by the functor*

$$\mathrm{Ct}_f : L_\Theta(\Phi) \to K_{\Phi\Theta}(f).$$

This functor Ct_f transforms knowledge description to content of knowledge. We do not assume that between different f_1 and f_2 in F there are any ties: instances are independent. On the other hand, between some f_1 and f_2 there may be relations that we will try to take into account.

A content of knowledge $\mathrm{Ct}_f(X, T) = (X, T^f)$ corresponds to an object (X, T) of the category $L_\Theta(\Phi)$, which is a description of knowledge. We view the description T as a *query* to a knowledge base, and $A = T^f$ as a *reply to this query.*

Besides, if there is a relation s_* between (X, T_1) and (Y, T_2), then there will be a relation $\widetilde{s} = \widetilde{s_*}$ between (X, A) and (Y, B), where $A = T_1^f, \ B = T_2^f$.

This peculiarity of the definition naturally reflects geometrical essence of knowledge.

In fact, in this definition of a knowledge base the category of knowledge is decomposed to two categories: the category of description of knowledge and the category of content of knowledge, tied by the functor of transition from description to content.

9 Equivalence of knowledge bases

9.1 Definition.

Let knowledge bases $KB_1 = KB(G_1, \Phi_1, F_1)$ and $KB_2 = KB(G_2, \Phi_2, F_2)$ correspond to the given multimodels (G_1, Φ_1, F_1) and (G_2, Φ_2, F_2).

Definition 1. *Knowledge bases KB_1 and KB_2 are called informationally equivalent, if there exists a bijection $\alpha : F_1 \to F_2$ such that for every $f \in F_1$ there exist homomorphisms*

$$\beta_f : \mathrm{Hal}_\Theta(\Phi_1) \to \mathrm{Hal}_\Theta(\Phi_2)$$

$$\beta'_f : \mathrm{Hal}_\Theta(\Phi_2) \to \mathrm{Hal}_\Theta(\Phi_1)$$

and an isomorphism of categories

$$\widetilde{\gamma}_f : K_{\Phi_1\Theta}(f) \to K_{\Phi_2\Theta}(f^\alpha)$$

such that the commutative diagrams of functors of categories hold:

$$\begin{tikzcd} L_\Theta(\Phi_1) \arrow[r, "\widetilde{\beta}_f"] \arrow[d, "\mathrm{Ct}_f"'] & L_\Theta(\Phi_2) \arrow[d, "\mathrm{Ct}_{f^\alpha}"] \\ K_{\Phi_1\Theta}(f) \arrow[r, "\widetilde{\gamma}_f"] & K_{\Phi_2\Theta}(f^\alpha) \end{tikzcd}$$

and

$$\begin{tikzcd} L_\Theta(\Phi_1) \arrow[d, "\mathrm{Ct}_f"'] & L_\Theta(\Phi_2) \arrow[l, "\widetilde{\beta}'_f"'] \arrow[d, "\mathrm{Ct}_{f^\alpha}"] \\ K_{\Phi_1\Theta}(f) & K_{\Phi_2\Theta}(f^\alpha) \arrow[l, "(\widetilde{\gamma}_f)^{-1}"'] \end{tikzcd}$$

Denote these diagrams by $*$ and $**$, respectively. Rewrite commutative diagrams for the object (X, T) of the category $L_\Theta(\Phi_1)$ in the form $(X, T^f)^{\widetilde{\gamma}_f} = (X, T^{\beta_f f^\alpha})$ and for the object (X, T) of the category $L_\Theta(\Phi_2)$ in the form $(X, T^{f^\alpha})^{\widetilde{\gamma}_f^{-1}} = (X, T^{\beta'_f f})$.

From this follows

$$(X, T^f) = (X, T^{\beta_f f^\alpha})^{\widetilde{(\gamma_f)}^{-1}},$$

$$(X, T^{f^\alpha}) = (X, T^{\beta'_f f})^{\widetilde{\gamma}_f},$$

The last means that everything which can be known from KB_1 can be also known from KB_2 and vice versa. Similar property holds for morphisms, i.e. connections between objects. Equivalence of knowledge bases we consider as a triple $(\alpha, *, **)$, where $\alpha : F_1 \to F_2$ is a bijection, while $*$ and $**$ define the corresponding diagrams for every $f \in F_1$.

The next proposition deals with the transition from knowledge bases to databases. Let R_f be the image of the homomorphism $\mathrm{Val}_f : \mathrm{Hal}_\Theta(\Phi) \to \mathrm{Hal}_\Theta(G)$.

Proposition 19. *If a bijection $\alpha : F_1 \to F_2$ determines equivalence of the bases KB_1 and KB_2 then for every $f \in F_1$ we have an isomorphism of Halmos algebras $\gamma_f : R_f \to R_{f^\alpha}$.*

Proof.

Proceed from the corresponding diagrams $*$ and $**$. Given set X, take a set T of one element $u \in \mathrm{Hal}_{\Phi_1\Theta}(X)$. In this case $T^f = \mathrm{Val}_f(u)$. We have $Ct_f(X,T) = (X, \mathrm{Val}_f(u))$,

$$(X, \mathrm{Val}_f(u))^{\widetilde{\gamma}_f} = Ct_{f^\alpha}(X, u^\beta) = (X, (u^\beta)^{f^\alpha}) = (X, \mathrm{Val}_{f^\alpha}(u^\beta)).$$

Hence, $\widetilde{\gamma}_f$ transfers $\mathrm{Val}_f(u)$ to $\mathrm{Val}_{f^\alpha}(u^\beta)$ for every u, which means that $\widetilde{\gamma}_f$ induces a mapping $\gamma_f : R_f \to R_{f^\alpha}$. It is a homomorphism since Val_f and β are homomorphisms of algebras, and it is an injection since every R_f is a simple algebra [Pl1].

Let now u_1 be an arbitrary element of $\mathrm{Hal}_{\Phi_2\Theta}(X)$. Then the second diagram gives

$$(X, \mathrm{Val}_{f^\alpha}(u_1))^{\widetilde{\gamma}_f^{-1}} = (X, \mathrm{Val}_f(u_1^{\beta'_f})),$$

and

$$(X, \mathrm{Val}_{f^\alpha}(u_1)) = (X, \mathrm{Val}_f(u_1^{\beta'_f}))^{\widetilde{\gamma}_f} = (X, \mathrm{Val}_{f^\alpha}(u))^{\widetilde{\gamma}_f},$$

where $u = u_1^{\beta'_f}$. This implies that $\gamma_f : R_f \to R_{f^\alpha}$ is a surjection. Hence, we have an isomorphism $\gamma_f : R_f \to R_{f^\alpha}$.

Proceed now from the isomorphism of Halmos algebras $\gamma_f : R_f \to R_{f^\alpha}$.

9.2 The case of finite models.

First of all it is clear that for finite models (G, Φ, F) the corresponding KB remains, in general, infinite.

We prove the following main

Theorem 18. *Let the given models be finite. Then the knowledge bases KB_1 and KB_2 are equivalent if and only if there exists a bijection $\alpha : F_1 \to F_2$ such that for every $f \in F_1$ there is an isomorphism $\gamma_f : R_f \to R_{f^\alpha}$.*

Proof.

In one direction the statement is always true. Let now $\gamma_f : R_f \to R_{f^\alpha}$ be an isomorphism for every $f \in F_1$. According to the proposition 3 (see also [PT])

we have the homomorphisms $\beta_f : \mathrm{Hal}_\Theta(\Phi_1) \to \mathrm{Hal}_\Theta(\Phi_2)$ and $\beta'_f : \mathrm{Hal}_\Theta(\Phi_2) \to \mathrm{Hal}_\Theta(\Phi_1)$ such that the diagrams

$$\begin{tikzcd}
\mathrm{Hal}_\Theta(\Phi_1) \arrow[r, "\beta_f"] \arrow[d, "\mathrm{Val}_f"'] & \mathrm{Hal}_\Theta(\Phi_2) \arrow[d, "\mathrm{Val}_{f^\alpha}"] \\
R_f \arrow[r, "\gamma^f"] & R_{f^\alpha}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Hal}_\Theta(\Phi_1) \arrow[d, "\mathrm{Val}_f"'] & \mathrm{Hal}_\Theta(\Phi_2) \arrow[l, "\beta'_f"'] \arrow[d, "\mathrm{Val}_{f^\alpha}"] \\
R_f & R_{f^\alpha} \arrow[l, "\gamma_f^{-1}"']
\end{tikzcd}$$

are commutative.

Simultaneously, there are functors

$$\widetilde{\beta}_f : L_\Theta(\Phi_1) \to L_\Theta(\Phi_2),$$

$$\widetilde{\beta}'_f : L_\Theta(\Phi_2) \to L_\Theta(\Phi_1).$$

It is left to define the isomorphism of categories $\widetilde{\gamma_f} : K_{\Phi_1\Theta}(f) \to K_{\Phi_2\Theta}(f^\alpha)$ such that the diagrams of the types $*$ and $**$ be commutative.

First we define $\widetilde{\gamma}_f$ on objects and then on morphisms. Take an object (X,T) of the category $L_\Theta(\Phi_1)$ for an arbitrary object (X, A) of the category $K_{\Phi_1\Theta}(f)$ with $T^f = A$. We have $Ct_f(X, T) = (X, T^f) = (X, A)$. Set

$$(X, A)^{\widetilde{\gamma_f}} = (X, T^f)^{\widetilde{\gamma_f}} = (X, \bigcap_{u \in T} \gamma_f \mathrm{Val}_f(u)) =$$

$$(X, \bigcap_{u \in T} \mathrm{Val}_{f^\alpha}(u^{\beta_f})) = (X, T^{\beta_f f^\alpha}).$$

We want to show that this definition does not depend on the choice of the set T with $T^f = A$. Consider first the case when $T_1^f = T_2^f = A$ and the sets T_1 and T_2 are finite. We have: $(X, T_1^f)^{\widetilde{\gamma_f}} = (X, T_1^{\beta_f f^\alpha})$ and $(X, T_2^f)^{\widetilde{\gamma_f}} = (X, T_2^{\beta_f f^\alpha})$.

We need to check that $T_1^{\beta_f f^\alpha} = T_2^{\beta_f f^\alpha}$. Indeed,

$$T_1^{\beta_f f^\alpha} = \bigcap_{u_1 \in T_1} \mathrm{Val}_{f^\alpha}(\beta_f u_1)) = \bigcap_{u_1 \in T_1} \gamma_f \mathrm{Val}_f(u_1)).$$

Since $\gamma_f : R_f \to R_{f^\alpha}$ is an isomorphism of algebras and T_1, T_2 are finite sets, we can rewrite the expression in the form

$$T_1^{\beta_f f^\alpha} = \gamma_f(\bigcap_{u_1 \in T_1} \mathrm{Val}_f(u_1)) = \gamma_f(\bigcap_{u_2 \in T_2} \mathrm{Val}_f(u_2)) = \bigcap_{u_2 \in T_2} \gamma_f \mathrm{Val}_f(u_2)) = T_2^{\beta_f f^\alpha}.$$

Passing to the general case we proceed from finite models. Every finite model is geometrically noetherian, i.e., if $A = T_1^f = T_2^f$, then in T_1 and T_2 one can find finite subsets T_{01} and T_{02} with ${T_{01}}^f = T_{02}^f = A$. Here, $T_{01}^{\beta_f f_\alpha} = T_{02}^{\beta_f f_\alpha}$. We have to verify that $T_1^{\beta_f f^\alpha} = T_2^{\beta_f f^\alpha}$ and $T_{01}^{\beta_f f_\alpha} = \bigcap_{u_1 \in T_1} \mathrm{Val}_{f^\alpha}(\beta_f u_1)$. We can take a finite subset T_{10} in T_1 such that $T_1^{\beta_f f^\alpha} = T_{10}^{\beta_f f^\alpha}$. Take the union of sets T_{10} and T_{01} and denote it by T_{001}. Then $T_{001}^f = A = T_1^f$, $T_1^{\beta_f f^\alpha} = T_{001}^{\beta_f f^\alpha}$. Analogously, for T_2 take T_{002} and $A = T_{001}^f = T_{002}^f$. Besides that,

$$T_1^{\beta_f f^\alpha} = T_{001}^{\beta_f f^\alpha} = T_2^{\beta_f f^\alpha}.$$

The equality $T_1^{\beta_f f^\alpha} = T_2^{\beta_f f^\alpha}$ gives commutativity of the diagram for objects.

Similarly, we build $\widetilde{\gamma}_f^{-1}$ having γ_f^{-1} and the equality $\widetilde{\gamma}_f^{-1} = \widetilde{\gamma_f^{-1}}$ holds.

Now let us pass to morphisms. Remind first of all that to every homomorphism $s : W(Y) \to W(X)$ there correspond

$$s_*^1 : \mathrm{Hal}_{\Phi_1\Theta}(Y) \to \mathrm{Hal}_{\Phi_1\Theta}(X)$$

$$s_*^2 : \mathrm{Hal}_{\Phi_2\Theta}(Y) \to \mathrm{Hal}_{\Phi_2\Theta}(X).$$

Let the objects (Y, T_2) and (X, T_1) be given in $L_\Theta(\Phi_1)$. Recall that s is admissible in respect to T_2 and T_1 if $s_*^1(u) \in T_1$ for every $u \in T_2$. Here $s_*^1 : (Y, T_2) \to (X, T_1)$ is a morphism. Proceed further from an arbitrary homomorphism $\beta : \mathrm{Hal}_\Theta(\Phi_1) \to \mathrm{Hal}_\Theta(\Phi_2)$. It had been proved that if s is admissible in respect to T_2 and T_1 then the same s is admissible in respect to T_2^β and T_1^β as well, i.e., $s_*^1(u) \in T_1^\beta$ for every $u \in T_2^\beta$. Hence, we have a morphism

$$\widetilde{\beta}(s_*^1) = s_*^2 : (Y, T_2^\beta) \to (X, T_1^\beta).$$

Take now $\beta = \beta_f$ and apply Ct_{f^α}:

$$\mathrm{Ct}_{f^\alpha}(s_*^2) : \mathrm{Ct}_{f^\alpha}(X, T_1^{\beta_X}) \to \mathrm{Ct}_{f^\alpha}(Y, T_2^{\beta_X}).$$

It can be rewritten as

$$\mathrm{Ct}_{f^\alpha}(s_*^2) : (X, T_1^{\beta_X f^\alpha}) \to (X, T_2^{\beta_Y f^\alpha})$$

or

$$\mathrm{Ct}_{f^\alpha}(s_*^2) : (X, T_1^f)^{\gamma_f} \to (Y, T_2^f)^{\gamma_f}.$$

Let now $T_1^f = A$, $T_2^f = B$. For $s_*^1 : (Y, T_2) \to (X, T_1)$ we have

$$\mathrm{Ct}_f(s_*^1) : (X, T_1^f) \to (Y, T_2^f)$$

and a related morphism

$$\mathrm{Ct}_{f^\alpha}(s_*^2) : (X, T_1^f)^{\gamma_f} \to (Y, T_2^{f_2})^{\gamma_f}$$

Commutativity of the diagram on morphisms means that

$$\widetilde{\gamma}_f \mathrm{Ct}_f(s_*^1) = \mathrm{Ct}_{f^\alpha}(\widetilde{\beta}_f(s_*^1))$$

for every $s_*^1 : (Y, T_2) \to (X, T_1)$.

Continuing consideration of finite models, proceed from the isomorphism $\gamma_f : R_f \to R_{f^\alpha}$ and the corresponding functor $\widetilde{\gamma}_f : K_{\Phi_1\Theta}(f) \to K_{\Phi_2\Theta}(f^\alpha)$. This functor had been defined on the objects, and now we are going to define it on morphisms.

Let $\tau : (X, A) \to (Y, B)$ be a morphism in $K_{\Phi_1\Theta}(f)$. This τ appears as follows. A morphism

$$s_*^1 : \mathrm{Hal}_{\Phi_1\Theta}(Y) \to \mathrm{Hal}_{\Phi_1\Theta}(X)$$

corresponds to $s : W(Y) \to W(X)$. If now $A = T_1^f$, $B = T_2^f$ and s_*^1 is admissible in respect to T_2 and T_1 then we have $\widetilde{s}_*^1 : (X, A) \to (Y, B)$. We may say that $\tau = \widetilde{s}_*^1$ for some s_*^1.

Define

$$\widetilde{\gamma}_f(\widetilde{s}_*^1) = \widetilde{s}_*^2 : (X, T_1^f)^{\widetilde{\gamma}_f} \to (Y, T_2^f)^{\widetilde{\gamma}_f}.$$

Here,

$$(X, T_1^f)^{\widetilde{\gamma}_f} = (X, T_1^{\beta_f f^\alpha}),$$

$$(Y, T_2^f)^{\widetilde{\gamma}_f} = (Y, T_2^{\beta_f f^\alpha}).$$

do not depend on the choice of T_1 and T_2 with $T_1^f = A$ and $T_2^f = B$. Check further that $\widetilde{\gamma}_f : K_{\Phi_1\Theta}(f) \to K_{\Phi_2\Theta}(f^\alpha)$ determined in such a way is in fact a functor and this functor provides commutativity of the diagram on morphisms.

Note first of all that the definition of $\widetilde{\gamma}_f$ on morphisms can be rewritten as

$$\widetilde{\gamma}_f(\mathrm{Ct}_f(s_*^1)) = (\mathrm{Ct}_{f^\alpha}(s_*^2)).$$

Take two morphisms $\widetilde{s^1_{1*}} = \mathrm{Ct}_f(s^1_{1*})$ and $\widetilde{s^1_{2*}} = \mathrm{Ct}_f(s^1_{2*})$ and consider the product

$$\widetilde{s^1_{1*}}\widetilde{s^1_{2*}} = \mathrm{Ct}_f(s^1_{1*})\mathrm{Ct}_f(s^1_{2*}) = \mathrm{Ct}_f(s^1_{2*}s^1_{1*}) = \widetilde{s^1_{2*}s^1_{1*}} = \widetilde{(s_2s_1)^1_*}.$$

Apply $\widetilde{\gamma}_f$:

$$\widetilde{\gamma}_f(\widetilde{(s_2s_1)^1_*}) = (\widetilde{(s_2s_1)^2_*}) = \widetilde{s^2_{2*}s^2_{1*}} = \widetilde{s^2_{1*}}\widetilde{s^2_{2*}} = \widetilde{\gamma}_f(\widetilde{s^1_{1*}})\widetilde{\gamma}_f(\widetilde{s^1_{2*}})$$

Now check the commutativity of the diagram

$$\begin{array}{ccc}
L_\Theta(\Phi_1) & \xrightarrow{\widetilde{\beta}_X} & L_\Theta(\Phi_2) \\
\mathrm{Ct}_f\downarrow & & \downarrow\mathrm{Ct}_{f^\alpha} \\
K_{\Phi_1\Theta}(f) & \xrightarrow{\widetilde{\gamma}_f} & K_{\Phi_1\Theta}(f^\alpha)
\end{array}$$

Take a morphism $s^1_* : (Y, T_2) \to (X, T_1)$ in $L_\Theta(\Phi_1)$. We have

$$\widetilde{\beta_X}(s^1_*) : (Y, T_2^{\beta_X}) \to (X, T_1^{\beta_X}),$$

and

$$\mathrm{Ct}_{f^\alpha}\widetilde{\beta_X}(s^1_*) : (X, T_1^{\beta_X f^\alpha}) \to (Y, T_2^{\beta_X f^\alpha}),$$

Rewrite it as

$$\mathrm{Ct}_{f^\alpha}\widetilde{\beta_X}(s^1_*) : (X, T_1^f)^{\widetilde{\gamma}_f} \to (Y, T_2^f)^{\widetilde{\gamma}_f},$$

Further,

$$\mathrm{Ct}_f(s^1_*) : (X, T_1^f) \to (Y, T_2^f),$$

$$\widetilde{\gamma}_f\mathrm{Ct}_f(s^1_*) : (X, T_1^f)^{\widetilde{\gamma}_f} \to (Y, T_2^f)^{\widetilde{\gamma}_f}.$$

Check now the equality

$$\widetilde{\gamma}_f\mathrm{Ct}_f(s^1_*) = \mathrm{Ct}_{f^\alpha}\widetilde{\beta}_X(s^1_*)$$

for every s^1_*. We have

$$\widetilde{\gamma}_f\mathrm{Ct}_f(s^1_*) = \widetilde{\gamma}_f(\widetilde{s}^1_*) = \widetilde{s}^2_*,$$

$$\mathrm{Ct}_{f^\alpha}\widetilde{\beta}_X(s^1_*) = \mathrm{Ct}_{f^\alpha}(s^2_*) = \widetilde{s}^2_*.$$

This gives commutativity of the diagram $*$ of morphisms, i.e.,

$$\widetilde{\gamma}_f\mathrm{Ct}_f = \mathrm{Ct}_{f^\alpha}\widetilde{\beta}_X.$$

The same can be done for the functor $\widetilde{\gamma_f^{-1}} = \widetilde{\gamma}_f^{-1}$ and the second commutative diagram $**$ that finishes the proof of the theorem

9.2 Main result. Additional remarks.

7.1. Let us look at the definition of equivalence from the general perspective of category theory. Given two functors $\varphi_1 : C_1 \to C_1^0$ and $\varphi_2 : C_2 \to C_2^0$, we say that C_1 and C_2 are equivalent in respect to φ_1 and φ_2, if there is an isomorphism $\psi : C_1^0 \to C_2^0$ and functors $\psi_1 : C_1 \to C_2$, $\psi_2 : C_2 \to C_1$ with the commutative diagrams

$$\begin{tikzcd} C_1 \arrow[r, "\psi_1"] \arrow[d, "\varphi_1"'] & C_2 \arrow[d, "\varphi_2"] \\ C_1^0 \arrow[r, "\psi"] & C_2^0 \end{tikzcd}$$

$$\begin{tikzcd} C_1 \arrow[d, "\varphi_1"'] & C_2 \arrow[l, "\psi_2"'] \arrow[d, "\varphi_2"] \\ C_1^0 & C_2^0 \arrow[l, "\psi^{-1}"'] \end{tikzcd}$$

Usual equivalence of categories is equivalence in respect to the transition to skeletons of categories. In our situation we may say that equivalence of knowledge bases means that there exists equivalence of categories of description of knowledge in respect to transition to the categories of knowledge content.

7.2. Return to the definition of knowledge bases with multimodels (G_1, Φ_1, F_1) and (G_2, Φ_2, F_2), and let the bijection $\alpha : F_1 \to F_2$ determine equivalence of the corresponding KB_1 and KB_2. Assume that two instances f_1 and f_2 from F_1 are connected by a commutative diagram

$$\begin{tikzcd} \mathrm{Hal}_\Theta(\Phi_1) \arrow[r, "\mathrm{Val}_{f_1}"] \arrow[dr, "\mathrm{Val}_{f_2}"'] & R_{f_1} \arrow[d, "\gamma"] \\ & R_{f_2} \end{tikzcd}$$

where γ is a homomorphism of algebras. We want to estimate the relation between f_1^α and f_2^α.

Proceed from the diagrams

$$\begin{tikzcd} \mathrm{Hal}_{\Phi_1\Theta} \arrow[r, "\beta_f"] \arrow[d, "\mathrm{Val}_f"'] & \mathrm{Hal}_{\Phi_2\Theta} \arrow[d, "\mathrm{Val}_{f^\alpha}"] \\ R_f \arrow[r, "\gamma_f"] & R_{f^\alpha} \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Hal}_{\Phi_1\Theta} \arrow[d, "\mathrm{Val}_f"'] & \mathrm{Hal}_{\Phi_2\Theta} \arrow[l, "\beta'_f"'] \arrow[d, "\mathrm{Val}_{f^\alpha}"] \\ R_f & R_{f^\alpha} \arrow[l, "\gamma_f^{-1}"'] \end{tikzcd}$$

$$\begin{tikzcd} R_{f_1} \arrow[r, "\gamma_{f_1}"] \arrow[d, "\gamma"'] & R_{f_1^\alpha} \arrow[d, "\gamma^\alpha"] \\ R_{f_2} \arrow[r, "\gamma_{f_2}"] & R_{f_2^\alpha} \end{tikzcd}$$

Here,

$$\gamma \mathrm{Val}_{f^1} = \mathrm{Val}_{f^2}, \qquad \gamma^\alpha = \gamma_{f_2}\gamma\gamma_{f_1}^{-1}$$

and

$$\gamma^\alpha \mathrm{Val}_{f_1^\alpha} = \gamma^\alpha \gamma_{f_1} \mathrm{Val}_{f_1} \beta'_{f_1} = \gamma_{f_2}\gamma \mathrm{Val}_{f_1}\beta'_{f_1} = \gamma_{f_2}\mathrm{Val}_{f_2}\beta'_{f_1} = \mathrm{Val}_{f_2^\alpha}\beta_{f_2}\beta'_{f_1},$$

Hence, $\gamma^\alpha \mathrm{Val}_{f^1\alpha} = \mathrm{Val}_{f_2^\alpha}\beta_{f_2}\beta'_{f_1}$, i.e., the connection is twisted by the product $\beta_{f_2}\beta'_{f_1}$.

At last, let us note that from the diagrams above follow the natural identities:

1. $\mathrm{Val}_f(u) = \mathrm{Val}_f(\beta'_f\beta_f(u))$ for every $u \in \mathrm{Hal}_\Theta(\Phi_1)$.
2. $\mathrm{Val}_{f^\alpha}(u) = \mathrm{Val}_{f^\alpha}(\beta_f\beta'_f(u)$ for every $u \in \mathrm{Hal}_\Theta(\Phi_2)$.

7.3. Note that the equivalence condition of two knowledge bases in the case of finite multimodels can be formulated in terms of these multimodels (cf. [PTP]).

Definition 2. *Let the models (G_1, Φ_1, f_1) and (G_2, Φ_2, f_2) be given. Let* $\mathrm{Aut}(f_1)$ *and* $\mathrm{Aut}(f_2)$ *be the corresponding groups of automorphisms. The models (G_1, Φ_1, f_1) and (G_2, Φ_2, f_2) are called automorphic equivalent if there exists an isomorphism of algebras $\delta : G_1 \to G_2$ such that*

$$\mathrm{Aut}(f_2) = \delta \mathrm{Aut}(f_1)\delta^{-1}.$$

Definition 3. *Let the multimodels (G_1, Φ_1, F_1) and (G_2, Φ_2, F_2) be given. These multimodels are called automorphic equivalent if there exists a bijection $\alpha : F_1 \to F_2$ such that for every $f \in F_1$ the models (G_1, Φ_1, f) and (G_2, Φ_2, f^α) are automorphic equivalent.*

It is natural to define an isomorphism of multimodels with the same set of relations Φ_1 and Φ_2. An isomorphism of multimodels implies their automorphic equivalence. Evidently, the inverse statement is not true.

Let the knowledge bases $KB_1 = KB(G_1, \Phi_1, F_1)$ and $KB_2 = KB(G_2, \Phi_2, F_2)$ with the finite multimodels be given.

Theorem 19. *The knowledge bases $KB_1 = KB(G_1, \Phi_1, F_1)$ and $KB_2 = KB(G_2, \Phi_2, F_2)$ are informationally equivalent if and only if the corresponding models are automorphic equivalent.*

The proof of this theorem uses the Galois-Krasner theory and follows from Theorems 14 and 18.

Theorem 19 implies an algorithm for the informational equivalence verification (see [PK]).

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